ook by a space, e.g. MJ m^{-3} not MJm^{-3}: this is not strictly necessary but is an id to clarity.

Finally it should be reemphasized that strain is simply a number. It is a imensionless quantity and is not expressed in physical units.

Example The shear stress required to nucleate a grain boundary crack in high-temperature deformation has been estimated to be

$$\tau = \left(\frac{3\pi\gamma_b G}{8(1-\nu)L} \right)^{1/2}$$

where γ_b is the grain boundary surface energy, let us say 2 J m^{-2}; G is the shear modulus, 75 GPa; L is the grain boundary sliding distance, assumed equal to the grain diameter 0.01 mm, and ν is Poisson's ratio, $\nu = 0.3$. To calculate τ we need to be sure the units are consistent and that the prefixes have been properly evaluated.

To check the equation express all units in newtons and meters.

$$\tau = \left(\frac{\frac{\text{N m}}{\text{m}^2} \times \frac{\text{N}}{\text{m}^2}}{\text{m}} \right)^{1/2} = \left(\frac{\text{N}^2}{\text{m}^4} \right)^{1/2} = \frac{\text{N}}{\text{m}^2}$$

Note that a joule (J) is a unit of energy; J = N m (see Appendix A)

$$\tau = \left(\frac{3\pi \times 2 \times 75 \times 10^9}{8(1-0.3) \times 10^{-2} \times 10^{-3}} \right)^{1/2} = (252.4 \times 10^{14})^{1/2}$$

$$= 15.89 \times 10^7 \text{ N m}^{-2}$$

$$= 158.9 \text{ MN m}^{-2} = 158.9 \text{ MPa}$$

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CHAPTER
TWO

STRESS AND STRAIN RELATIONSHIPS FOR ELASTIC BEHAVIOR

2-1 INTRODUCTION

The purpose of this chapter is to present the mathematical relationships for expressing the stress and strain at a point and the relationships between stress and strain in a solid which obeys Hooke's law. While part of the material covered in this chapter is a review of information generally covered in strength of materials, the subject is extended beyond this point to a consideration of stress and strain in three dimensions. The material included in this chapter is important for an understanding of most of the phenomenological aspects of mechanical metallurgy, and for this reason it should be given careful attention by those readers to whom it is unfamiliar. In the space available for this subject it has not been possible to carry it to the point where extensive problem solving is possible. The material covered here should, however, provide a background for intelligent reading of the more mathematical literature in mechanical metallurgy.

It should be recognized that the equations describing the state of stress or strain in a body are applicable to any solid continuum, whether it be an elastic or plastic solid or a viscous fluid. Indeed, this body of knowledge is often called *continuum mechanics*. The equations relating stress and strain are called *constitutive equations* because they depend on the material behavior. In this chapter we shall only consider the constitutive equations for an elastic solid.

2-2 DESCRIPTION OF STRESS AT A POINT

As described in Sec. 1-8, it is often convenient to resolve the stresses at a point into normal and shear components. In the general case the shear components are at arbitrary angles to the coordinate axes, so that it is convenient to resolve each

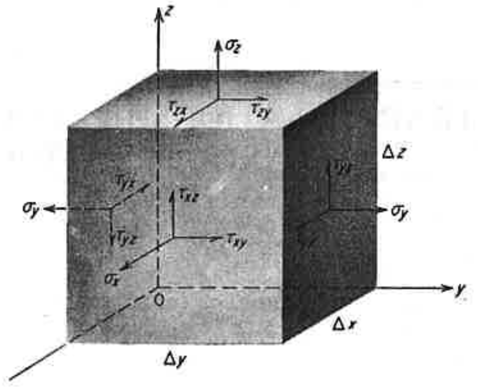


Figure 2-1 Stresses acting on an elemental cube.

ıear stress further into two components. The general case is shown in Fig. 2-1. resses acting normal to the faces of the elemental cube are identified by the ıbscript which also identifies the direction in which the stress acts; that is σ_x is e normal stress acting in the x direction. Since it is a normal stress, it must act ı the plane perpendicular to the x direction. By convention, values of normal resses greater than zero denote tension; values less than zero indicate compres- on. All the normal stresses shown in Fig. 2-1 are tensile.

Two subscripts are needed for describing shearing stresses. The first subscript dicates the plane in which the stress acts and the second the direction in which ıe stress acts. Since a plane is most easily defined by its normal, the first ıbscript refers to this normal. For example, τ_{yz} is the shear stress on the plane erpendicular to the y axis in the direction of the z axis. τ_{yx} is the shear stress on plane normal to the y axis in the direction of the x axis.

A shear stress is positive if it points in the positive direction on the positive ce of a unit cube. (It is also positive if it points in the negative direction on the gative face of a unit cube.) All of the shear stresses in Fig. 2-2a are positive ıear stresses regardless of the type of normal stresses that are present. A shear ress is negative if it points in the negative direction of a positive face of a unit ıbe and vice versa. The shearing stresses shown in Fig. 2-2b are all negative resses.

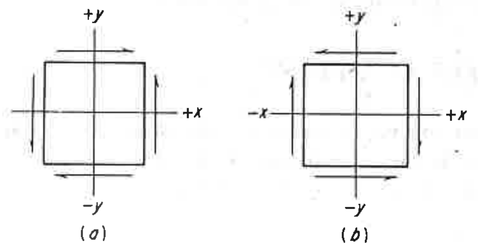


Figure 2-2 Sign convention for shear stress. (a) Positive; (b) negative.

The notation for stress given above is the one used by Timoshenko[1] and most American workers in the field of elasticity. However, many other notations have been used, some of which are given below.

$$\begin{array}{ccccc}
\sigma_x & \sigma_{11} & X_x & \widehat{xx} & p_{xx} \\
\sigma_y & \sigma_{22} & Y_y & \widehat{yy} & p_{yy} \\
\sigma_z & \sigma_{33} & Z_z & \widehat{zz} & p_{zz} \\
\tau_{xy} & \sigma_{12} & X_y & \widehat{xy} & p_{xy} \\
\tau_{yz} & \sigma_{23} & Y_z & \widehat{yz} & p_{yz} \\
\tau_{zx} & \sigma_{31} & Z_x & \widehat{zx} & p_{zx}
\end{array}$$

It can be seen from Fig. 2-1 that nine quantities must be defined in order to establish the state of stress at a point. They are σ_x, σ_y, σ_z, τ_{xy}, τ_{xz}, τ_{yx}, τ_{yz}, τ_{zx}, and τ_{zy}. However, some simplification is possible. If we assume that the areas of the faces of the unit cube are small enough so that the change in stress over the face is negligible, by taking the summation of the moments of the forces about the z axis it can be shown that $\tau_{xy} = \tau_{yx}$.

$$(\tau_{xy}\,\Delta y \Delta z)\,\Delta x = (\tau_{yx}\,\Delta x\,\Delta z)\,\Delta y \qquad (2\text{-}1)$$
$$\therefore \quad \tau_{xy} = \tau_{yx}$$

and in like manner

$$\tau_{xz} = \tau_{zx} \qquad \tau_{yz} = \tau_{zy}$$

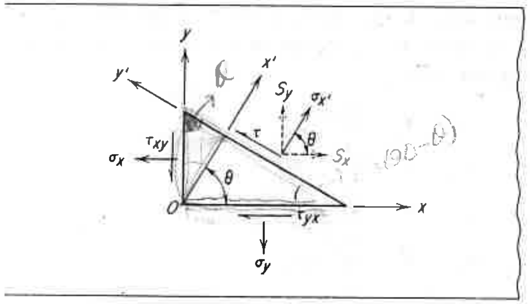
Thus, the state of stress at a point is completely described by six components: three normal stresses and three shear stresses, σ_x, σ_y, σ_z, τ_{xy}, τ_{xz}, τ_{yz}.

2-3 STATE OF STRESS IN TWO DIMENSIONS (PLANE STRESS)

Many problems can be simplified by considering a two-dimensional state of stress. This condition is frequently approached in practice when one of the dimensions of the body is small relative to the others. For example, in a thin plate loaded in the plane of the plate there will be no stress acting perpendicular to the surface of the plate. The stress system will consist of two normal stresses σ_x and σ_y and a shear stress τ_{xy}. A stress condition in which the stresses are zero in one of the primary directions is called *plane stress*.

Figure 2-3 illustrates a thin plate with its thickness normal to the plane of the paper. In order to know the state of stress at point O in the plate, we need to be able to describe the stress components at O for any orientation of the axes through the point. To do this, consider an oblique plane normal to the plane of the paper at an angle θ between the x axis and the outward normal to the oblique plane. Let the normal to this plane be the x' direction and the direction lying in

[1] S. P. Timoshenko, and J. N. Goodier, "Theory of Elasticity," 2d ed., McGraw-Hill Book Company, New York, 1951.



A.cos(90−θ)

Figure 2-3 Stress on oblique plane (two dimensions).

ıe oblique plane the y' direction. It is assumed that the plane shown in Fig. 2-3 ; an infinitesimal distance from O and that the element is so small that variations ı stress over the sides of the element can be neglected. The stresses acting on the blique plane are the normal stress σ and the shear stress τ. The direction cosines etween x' and the x and y axes are l and m, respectively. From the geometry of 'ig. 2-3, $l = \cos\theta$ and $m = \sin\theta$. If A is the area of the oblique plane, the areas f the sides of the element perpendicular to the x and y axes are Al and Am.

Let S_x and S_y denote the x and y components of the total stress acting on ıe inclined face. By taking the summation of the *forces* in the x direction and ıe y direction, we obtain

A.cosθ A.sinθ

x-function $S_xA = \sigma_x Al + \tau_{xy} Am$ τ_{yx}

y function $S_yA = \sigma_y Am + \tau_{xy} Al$

r

$$S_x = \sigma_x \cos\theta + \tau_{xy} \sin\theta$$

$$S_y = \sigma_y \sin\theta + \tau_{xy} \cos\theta$$

he components of S_x and S_y in the direction of the normal stress σ are

$$S_{xN} = S_x \cos\theta \quad \text{and} \quad S_{yN} = S_y \sin\theta$$

› that the normal stress acting on the oblique plane is given by

$$\sigma_{x'} = S_x \cos\theta + S_y \sin\theta$$

$$\sigma_{x'} = \sigma_x \cos^2\theta + \sigma_y \sin^2\theta + 2\tau_{xy} \sin\theta\cos\theta \qquad (2\text{-}2)$$

he shearing stress on the oblique plane is given by

$$\tau_{x'y'} = S_y \cos\theta - S_x \sin\theta$$

$$\tau_{x'y'} = \tau_{xy}(\cos^2\theta - \sin^2\theta) + (\sigma_y - \sigma_x)\sin\theta\cos\theta \qquad (2\text{-}3)$$

he stress $\sigma_{y'}$ may be found by substituting $\theta + \pi/2$ for θ in Eq. (2-2), since $\sigma_{y'}$ orthogonal to $\sigma_{x'}$.

$$_{'} = \sigma_x \cos^2(\theta + \pi/2) + \sigma_y \sin^2(\theta + \pi/2) + 2\tau_{xy}\sin(\theta + \pi/2)\cos(\theta + \pi/2)$$

ıd since $\sin(\theta + \pi/2) = \cos\theta$ and $\cos(\theta + \pi/2) = -\sin\theta$, we obtain

$$\sigma_{y'} = \sigma_x \sin^2\theta + \sigma_y \cos^2\theta - 2\tau_{xy}\sin\theta\cos\theta \qquad (2\text{-}4)$$

Equations (2-2) to (2-4) are the transformation of stress equations which give the stresses in an $x'y'$ coordinate system if the stresses in an xy coordinate system and the angle θ are known.

To aid in computation, it is often convenient to express Eqs. (2-2) to (2-4) in terms of the double angle 2θ. This can be done with the following identities:

$$\cos^2\theta = \frac{\cos 2\theta + 1}{2}$$

$$\sin^2\theta = \frac{1 - \cos 2\theta}{2}$$

$$2\sin\theta\cos\theta = \sin 2\theta$$

$$\cos^2\theta - \sin^2\theta = \cos 2\theta$$

The transformation of stress equations now become

$$\sigma_{x'} = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2}\cos 2\theta + \tau_{xy}\sin 2\theta \qquad (2\text{-}5)$$

$$\sigma_{y'} = \frac{\sigma_x + \sigma_y}{2} - \frac{\sigma_x - \sigma_y}{2}\cos 2\theta - \tau_{xy}\sin 2\theta \qquad (2\text{-}6)$$

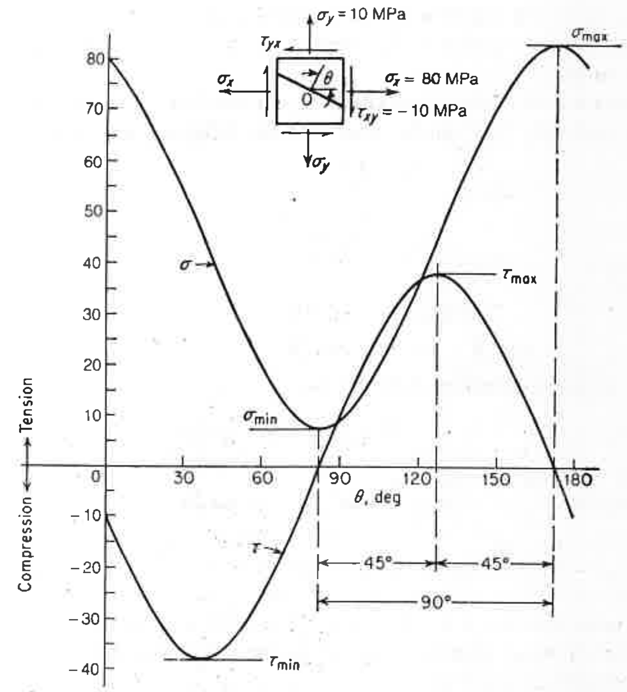
$$\tau_{x'y'} = \frac{\sigma_y - \sigma_x}{2}\sin 2\theta + \tau_{xy}\cos 2\theta \qquad (2\text{-}7)$$

It is important to note that $\sigma_{x'} + \sigma_{y'} = \sigma_x + \sigma_y$. Thus the sum of the normal stresses on two perpendicular planes is an *invariant* quantity, that is, it is independent of orientation or angle θ.

Equations (2-2) and (2-3) and their equivalents, Eqs. (2-5) and (2-7), describe the normal stress and shear stress on any plane through a point in a body subjected to a plane-stress situation. Figure 2-4 shows the variation of normal stress and shear stress with θ for the biaxial-plane-stress situation given at the top of the figure. Note the following important facts about this figure:

1. The maximum and minimum values of normal stress on the oblique plane through point O occur when the shear stress is zero.
2. The maximum and minimum values of both normal stress and shear stress occur at angles which are 90° apart.
3. The maximum shear stress occurs at an angle halfway between the maximum and minimum normal stresses.
4. The variation of normal stress and shear stress occurs in the form of a sine wave, with a period of $\theta = 180°$. These relationships are valid for any state of stress.

For any state of stress it is always possible to define a new coordinate system which has axes perpendicular to the planes on which the maximum normal stresses act and on which no shearing stresses act. These planes are called the *principal planes*, and the stresses normal to these planes are the *principal stresses*. For two-dimensional plane stress there will be two principal stresses σ_1 and σ_2 which occur at angles that are 90° apart (Fig. 2-4). For the general case of stress



ure 2-4 Variation of normal stress and shear stress on oblique plane with angle θ.

three dimensions there will be three principal stresses σ_1, σ_2, and σ_3. According convention, σ_1 is the algebraically greatest principal stress, while σ_3 is the gebraically smallest stress. The directions of the principal stresses are the *incipal axes* 1, 2, and 3. Although in general the principal axes 1, 2, and 3 do t coincide with the cartesian-coordinate axes x, y, z, for many situations that e encountered in practice the two systems of axes coincide because of symmetry loading and deformation. The specification of the principal stresses and their rection provides a convenient way of describing the state of stress at a point.

Since by definition a principal plane contains no shear stress, its angular lationship with respect to the xy coordinate axes can be determined by finding e values of θ in Eq. (2-3) for which $\tau_{x'y'} = 0$.

$$\tau_{xy}(\cos^2\theta - \sin^2\theta) + (\sigma_y - \sigma_x)\sin\theta\cos\theta = 0$$

$$\frac{\tau_{xy}}{\sigma_x - \sigma_y} = \frac{\sin\theta\cos\theta}{\cos^2\theta - \sin^2\theta} = \frac{\frac{1}{2}(\sin 2\theta)}{\cos 2\theta} = \frac{1}{2}\tan 2\theta$$

$$\tan 2\theta = \frac{2\tau_{xy}}{\sigma_x - \sigma_y} \tag{2-8}$$

Since $\tan 2\theta = \tan(\pi + 2\theta)$, Eq. (2-8) has two roots, θ_1 and $\theta_2 = \theta_1 + n\pi/2$. These roots define two mutually perpendicular planes which are free from shear.

Equation (2-5) will give the principal stresses when values of $\cos 2\theta$ and $\sin 2\theta$ are substituted into it from Eq. (2-8). The values of $\cos 2\theta$ and $\sin 2\theta$ are found from Eq. (2-8) by means of the pythagorean relationships.

$$\sin 2\theta = \pm\frac{\tau_{xy}}{\left[(\sigma_x - \sigma_y)^2/4 + \tau_{xy}^2\right]^{1/2}}$$

$$\cos 2\theta = \pm\frac{(\sigma_x - \sigma_y)/2}{\left[(\sigma_x - \sigma_y)^2/4 + \tau_{xy}^2\right]^{1/2}}$$

Substituting these values into Eq. (2-5) results in the expression for the maximum and minimum principal stresses for a two-dimensional (biaxial) state of stress.

$$\left.\begin{aligned}\sigma_{max} &= \sigma_1\\ \sigma_{min} &= \sigma_2\end{aligned}\right\} = \frac{\sigma_x + \sigma_y}{2} \pm \left[\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2\right]^{1/2} \tag{2-9}$$

The direction of the principal planes is found by solving for θ in Eq. (2-8), taking special care to establish whether 2θ is between 0 and $\pi/2$, π, and $3\pi/2$, etc. Figure 2-5 shows a simple way to establish the direction of the largest principal stress σ_1. σ_1 will lie between the algebraically largest normal stress and the shear diagonal. To see this intuitively, consider that if there were no shear stresses, then $\sigma_x = \sigma_1$. If only shear stresses act, then a normal stress (the principal stress) would exist along the shear diagonal. If both normal and shear stresses act on the element, then σ_1 lies between the influences of these two effects.

To find the maximum shear stress we return to Eq. (2-7). We differentiate the expression for $\tau_{x'y'}$ and set this equal to zero.

$$\frac{d\tau_{x'y'}}{d\theta} = (\sigma_y - \sigma_x)\cos 2\theta - 2\tau_{xy}\sin 2\theta = 0$$

$$\tan 2\theta_s = \frac{\sigma_y - \sigma_x}{2\tau_{xy}} = -\frac{\sigma_x - \sigma_y}{2\tau_{xy}} \tag{2-10}$$

Comparing this with the angle at which the principal planes occur, Eq. (2-8), $\tan 2\theta_n = 2\tau_{xy}/(\sigma_x - \sigma_y)$, we see that $\tan 2\theta_s$ is the negative reciprocal of $\tan 2\theta_n$.

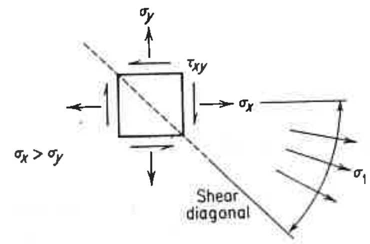


Figure 2-5 Method of establishing direction of σ_1.

ıis means that $2\theta_s$ and $2\theta_n$ are orthogonal, and that θ_s and θ_n are separated in ıace by 45°. The magnitude of the maximum shear stress is found by substitut-g Eq. (2-10) into Eq. (2-7).

$$\tau_{\max} = \pm\left[\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2\right]^{1/2} \tag{2-11}$$

Example The state of stress is given by $\sigma_x = 25p$ and $\sigma_y = 5p$ plus shearing stresses τ_{xy}. On a plane at 45° counterclockwise to the plane on which σ_x acts the state of stress is 50 MPa tension and 5 MPa shear. Determine the values of $\sigma_x, \sigma_y, \tau_{xy}$.

From Eqs. (2-5) and (2-7)

$$\sigma_{x'} = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2}\cos 2\theta + \tau_{xy}\sin 2\theta \qquad \text{Eq. (2-5)}$$

$$50 \times 10^6 = \frac{25p + 5p}{2} + \frac{25p - 5p}{2}\cos 90° + \tau_{xy}\sin 90°$$

$$15p + \tau_{xy} = 50 \times 10^6 \text{ Pa}$$

$$\tau_{x'y'} = \frac{\sigma_y - \sigma_x}{2}\sin 2\theta + \tau_{xy}\cos 2\theta \qquad \text{Eq. (2-7)}$$

$$5 \times 10^6 = \left(\frac{5p - 25p}{2}\right)\sin 90° + \tau_{xy}\cos 90$$

$$-10p = 5 \times 10^6 \qquad p = -5 \times 10^5 \text{ Pa}$$

$$\therefore \quad \sigma_x = 25(-5 \times 10^5) = -12.5 \text{ MPa}$$

$$\sigma_y = 5(p) = -2.5 \text{ MPa}$$

$$\tau_{xy} = 50 \times 10^6 - 15(-5 \times 10^5)$$

$$= 50 \times 10^6 + 7.5 \times 10^6 = 57.5 \text{ MPa}$$

We also can find $\sigma_{y'}$, orthogonal to $\sigma_{x'} = 50$ MPa, since $\sigma_x + \sigma_y = \sigma_{x'} + \sigma_{y'}$

$$-12.5 - 2.5 = 50 + \sigma_{y'}$$

$$\sigma_{y'} = -65 \text{ MPa}$$

4 MOHR'S CIRCLE OF STRESS—TWO DIMENSIONS

very useful graphical method for representing the state of stress at a point on oblique plane through the point was suggested by O. Mohr. The transforma-

tion of stress equations, Eqs. (2-5) and (2-7), can be rearranged to give

$$\sigma_{x'} - \frac{\sigma_x + \sigma_y}{2} = \frac{\sigma_x - \sigma_y}{2}\cos 2\theta + \tau_{xy}\sin 2\theta$$

$$\tau_{y'x'} = \frac{\sigma_y - \sigma_x}{2}\sin 2\theta + \tau_{xy}\cos 2\theta$$

We can solve for $\sigma_{x'}$ in terms of $\tau_{x'y'}$ by squaring each of these equations and adding

$(x - h)^2 + y^2 \qquad \tau_{max}^2$

$$\left(\sigma_{x'} - \frac{\sigma_x + \sigma_y}{2}\right)^2 + \tau_{x'y'}^2 = \left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2 \tag{2-12}$$

Equation (2-12) is the equation of a circle of the form $(x - h)^2 + y^2 = r^2$. Thus, Mohr's circle is a circle in $\sigma_{x'}, \tau_{x'y'}$ coordinates with a radius equal to $\tau_{\max}$ and the center displaced $(\sigma_x + \sigma_y)/2$ to the right of the origin.

In working with Mohr's circle there are only a few basic rules to remember. An angle of θ on the physical element is represented by 2θ on Mohr's circle. The same sense of rotation (clockwise or counterclockwise) should be used in each case. A different convention to express shear stress is used in drawing and interpreting Mohr's circle. This convention is that a shear stress causing a clockwise rotation about any point in the physical element is plotted above the horizontal axis of the Mohr's circle. A point on Mohr's circle gives the magnitude and direction of the normal and shear stresses on any plane in the physical element.

Figure 2-6 illustrates the plotting and use of Mohr's circle for a particular stress state shown at the upper left. Normal stresses are plotted along the x axis, shear stresses along the y axis. The stresses on the planes normal to the x and y

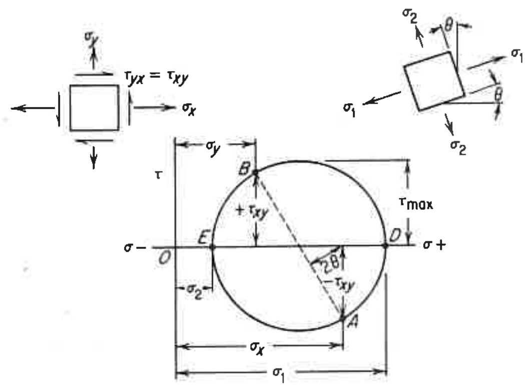


Figure 2-6 Mohr's circle for two-dimensional state of stress.

es are plotted as points A and B. The intersection of the line AB with the σ is determines the center of the circle. At points D and E the shear stress is ro, so these points represent the values of the principal stresses. The angle tween σ_x and σ_1 on Mohr's circle is 2θ. Since this angle is measured counter-ockwise on Mohr's circle on the physical element, σ_1 acts counterclockwise from e x axis at an angle θ (see sketch, upper right). The stresses on any other plane nose normal makes an angle of θ with the x axis could be found from Mohr's rcle in the same way.

5 STATE OF STRESS IN THREE DIMENSIONS

ie general three-dimensional state of stress consists of three unequal principal esses acting at a point. This is called a *triaxial state of stress*. If two of the three incipal stresses are equal, the state of stress is known as *cylindrical*, while if all ree principal stresses are equal, the state of stress is said to be *hydrostatic*, or *herical*.

The determination of the principal stresses for a three-dimensional state of ess in terms of the stresses acting on an arbitrary cartesian-coordinate system is extension of the method described in Sec. 2-3 for the two-dimensional case. gure 2-7 represents an elemental free body similar to that shown in Fig. 2-1 th a diagonal plane JKL of area A. The plane JKL is assumed to be a incipal plane cutting through the unit cube. σ is the principal stress acting rmal to the plane JKL. Let l, m, n be the direction cosines of σ, that is, the sines of the angles between σ and the x, y, and z axes. Since the free body in g. 2-7, must be in equilibrium, the forces acting on each of its faces must balance. The components of σ along each of the axes are S_x, S_y, and S_z.

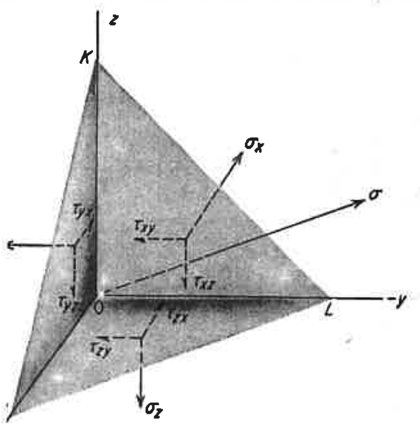


Figure 2-7 Stresses acting on elemental free body.

$$S_x = \sigma l \qquad S_y = \sigma m \qquad S_z = \sigma n$$

$$\text{Area } KOL = Al \qquad \text{Area } JOK = Am \qquad \text{Area } JOL = An$$

Taking the summation of the forces in the x direction results in

$$\sigma Al - \sigma_x Al - \tau_{yx} Am - \tau_{zx} An = 0$$

which reduces to

$$(\sigma - \sigma_x)l - \tau_{yx} m - \tau_{zx} n = 0 \tag{2-13a}$$

Summing the forces along the other two axes results in

$$-\tau_{xy} l + (\sigma - \sigma_y)m - \tau_{zy} n = 0 \tag{2-13b}$$

$$-\tau_{xz} l - \tau_{yz} m + (\sigma - \sigma_z)n = 0 \tag{2-13c}$$

Equations (2-13) are three homogeneous linear equations in terms of l, m, and n. The only nontrivial solution can be obtained by setting the determinant of the coefficients of l, m, and n equal to zero, since l, m, and n cannot all be zero.

$$\begin{vmatrix} \sigma - \sigma_x & -\tau_{yx} & -\tau_{zx} \\ -\tau_{xy} & \sigma - \sigma_y & -\tau_{zy} \\ -\tau_{xz} & -\tau_{yz} & \sigma - \sigma_z \end{vmatrix} = 0$$

Solution of the determinant results in a cubic equation in σ.

$$\sigma^3 - (\sigma_x + \sigma_y + \sigma_z)\sigma^2 + \left(\sigma_x\sigma_y + \sigma_y\sigma_z + \sigma_x\sigma_z - \tau_{xy}^2 - \tau_{yz}^2 - \tau_{xz}^2\right)\sigma$$
$$-\left(\sigma_x\sigma_y\sigma_z + 2\tau_{xy}\tau_{yz}\tau_{xz} - \sigma_x\tau_{yz}^2 - \sigma_y\tau_{xz}^2 - \sigma_z\tau_{xy}^2\right) = 0 \tag{2-14}$$

The three roots of Eq. (2-14) are the three principal stresses σ_1, σ_2, and σ_3. To determine the direction, with respect to the original x, y, z axes, in which the principal stresses act, it is necessary to substitute, σ_1, σ_2, and σ_3 each in turn into the three equations of Eq. (2-13). The resulting equations must be solved simultaneously for l, m, and n with the help of the auxiliary relationship $l^2 + m^2 + n^2 = 1$.

Note that there are three combinations of stress components in Eq. (2-14) that make up the coefficients of the cubic equation. Since the values of these coefficients determine the principal stresses, they obviously do not vary with changes in the coordinate axes. Therefore, they are invariant coefficients.

$$\sigma_x + \sigma_y + \sigma_z = I_1$$

$$\sigma_x\sigma_y + \sigma_y\sigma_z + \sigma_x\sigma_z - \tau_{xy}^2 - \tau_{xz}^2 - \tau_{yz}^2 = I_2$$

$$\sigma_x\sigma_y\sigma_z + 2\tau_{xy}\tau_{yz}\tau_{xz} - \sigma_x\tau_{yz}^2 - \sigma_y\tau_{xz}^2 - \sigma_z\tau_{xy}^2 = I_3$$

The first invariant of stress I_1 has been seen before for the two-dimensional state of stress. It states the useful relationship that the sum of the normal stresses for any orientation in the coordinate system is equal to the sum of the normal stresses or any other orientation. For example

$$\sigma_x + \sigma_y + \sigma_z = \sigma_{x'} + \sigma_{y'} + \sigma_{z'} = \sigma_1 + \sigma_2 + \sigma_3 \tag{2-15}$$

Example Determine the principal stresses for the state of stress

$$\begin{bmatrix} 0 & -240 & 0 \\ -240 & 200 & 0 \\ 0 & 0 & -280 \end{bmatrix} \text{MPa}$$

From Eq. (2-14)

$$\overset{I_1}{\sigma^3 - (200 - 280)\sigma^2} + \overset{I_2}{[(200)(-280) - (-240)^2]}\sigma - \overset{I_3}{(-280)(-240)^2} = 0$$

$\sigma = -280$ MPa is a principal stress because $\tau_{zx} = \tau_{xz} = 0$ and $\tau_{zy} = \tau_{yz} = 0$

$$[\sigma - (-280)](\sigma^3 - I_1\sigma^2 + I_2\sigma - I_3) = (\sigma^2 - 200\sigma - (240)^2) = 0$$

$$\sigma = \frac{200 \pm [(-200)^2 + 4(240)^2]^{1/2}}{2} = 100 \pm 260$$

$$\sigma_1 = 360 \text{ MPa}; \qquad \sigma_2 = -160 \text{ MPa}; \qquad \sigma_3 = -280 \text{ MPa}$$

In the discussion above we developed the equation for the stress on a articular oblique plane, a principal plane in which there is no shear stress. Let us ow develop the equations for the normal and shear stress on *any* oblique plane hose normal has the direction cosines l, m, n with the x, y, z axes. We can use ig. 2-7 once again if we realize that for this general situation the total stress on e plane S will not be coaxial with the normal stress, and that $S^2 = \sigma^2 + \tau^2$. nce again the total stress can be resolved into components S_x, S_y, S_z, so that

$$S^2 = S_x^2 + S_y^2 + S_z^2 \tag{2-16}$$

aking the summation of the *forces* in the x, y, and z directions, we arrive at the pressions for the orthogonal components of the total stress:

$$S_x = \sigma_x l + \tau_{yx} m + \tau_{zx} n \tag{2-17a}$$

$$S_y = \tau_{xy} l + \sigma_y m + \tau_{zy} n \tag{2-17b}$$

$$S_z = \tau_{xz} l + \tau_{yz} m + \sigma_z n \tag{2-17c}$$

To find the normal stress σ on the oblique plane, it is necessary to determine e components of S_x, S_y, S_z in the direction of the normal to the oblique plane. nus,

$$\sigma = S_x l + S_y m + S_z n$$

, after substituting from Eqs. (2-17) and simplifying with $\tau_{xy} = \tau_{yx}$, etc.

$$\sigma = \sigma_x l^2 + \sigma_y m^2 + \sigma_z n^2 + 2\tau_{xy} lm + 2\tau_{yz} mn + 2\tau_{zx} nl \tag{2-18}$$

The magnitude of the shear stress on the oblique plane can be found from $\tau^2 = S^2 - \sigma^2$. To get the magnitude and direction of the two shear stress components lying in the oblique plane it is necessary to resolve the stress components S_x, S_y, S_z into the y' and z' directions lying in the oblique plane.[1] This development will not be carried out here because the pertinent equations can be derived more easily by the methods given in Sec. 2-6.

Since plastic flow involves shearing stresses, it is important to identify the planes on which the *maximum* or *principal shear stresses* occur. In our discussion of the two-dimensional state of stress we saw that τ_{max} occurred on a plane halfway between the two principal planes. Therefore it is easiest to define the principal shear planes in terms of the three principal axes 1, 2, 3. From $\tau^2 = S^2 - \sigma^2$ it can be shown that

$$\tau^2 = (\sigma_1 - \sigma_2)^2 l^2 m^2 + (\sigma_1 - \sigma_3)^2 l^2 n^2 + (\sigma_2 - \sigma_3)^2 m^2 n^2 \tag{2-19}$$

where l, m, n are the direction cosines between the normal to the oblique plane and the principal axes.

The principal shear stresses occur for the following combinations of direction cosines that bisect the angle between two of the three principal axes:

	l	m	n	τ
l	0	$\pm\sqrt{\frac{1}{2}}$	$\pm\sqrt{\frac{1}{2}}$	$\tau_1 = \frac{\sigma_2 - \sigma_3}{2}$
m	$\pm\sqrt{\frac{1}{2}}$	0	$\pm\sqrt{\frac{1}{2}}$	$\tau_2 = \frac{\sigma_1 - \sigma_3}{2}$
n	$\pm\sqrt{\frac{1}{2}}$	$\pm\sqrt{\frac{1}{2}}$	0	$\tau_3 = \frac{\sigma_1 - \sigma_2}{2}$

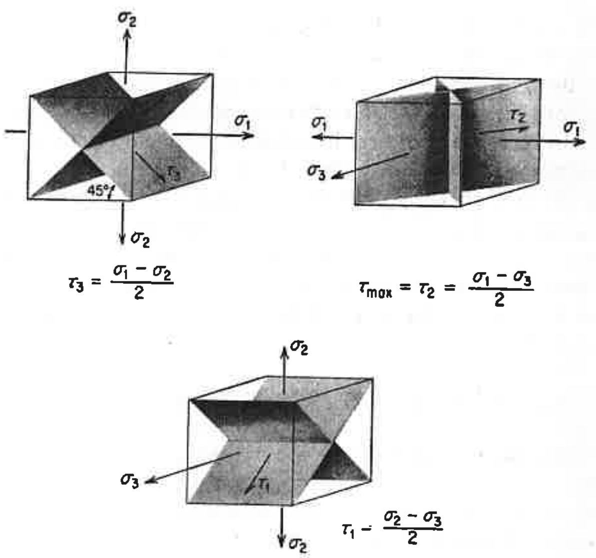
(2-20)

Since according to convention σ_1 is the algebraically greatest principal normal stress and σ_3 is the algebraically smallest principal stress, τ_2 has the largest value of shear stress and it is called the *maximum shear stress* τ_{max}.

$$\tau_{max} = \frac{\sigma_1 - \sigma_3}{2} \tag{2-21}$$

The maximum shear stress is important in theories of yielding and metal-forming operations. Figure 2-8 shows the planes of the principal shear stresses for a cube whose faces are the principal planes. Note that for each pair of principal stresses there are two planes of principal shear stress, which bisect the directions of the principal stresses.

[1] P. C. Chou and N. J. Pagano, "Elasticity," p. 24, D. Van Nostrand Company, Inc., Princeton, N.J., 1967.



ıure 2-8 Planes of principal shear stresses.

6 STRESS TENSOR

any aspects of the analysis of stress, such as the equations for the transforma-›n of the stress components from one set of coordinate axes to another ordinate system or the existence of principal stresses, become simpler when it is ılized that stress is a second-rank tensor quantity. Many of the techniques for ınipulating second-rank tensors do not require a deep understanding of tensor lculus, so it is advantageous to learn something about the properties of tensors.

We shall start with the consideration of the transformation of a vector (a st-rank tensor) from one coordinate system to another. Consider the vector $= S_1 i_1 + S_2 i_2 + S_3 i_3$, when the unit vectors i_1, i_2, i_3 are in the directions $, x_2, x_3$. (In accordance with convention and convenience in working with ısor quantities, the coordinate axes will be designated x_1, x_2, etc., where x_1 is uivalent to our previous designation x, x_2 is equivalent to the old y, etc.) $, S_2, S_3$ are the components of **S** referred to the axes x_1, x_2, x_3. We now want find the components of **S** referred to the x_1', x_2', x_3' axes, Fig. 2-9. S_1' is tained by resolving S_1, S_2, S_3 along the new direction x_1'.

$$S_1' = S_1 \cos(x_1 x_1') + S_2 \cos(x_2 x_1') + S_3 \cos(x_3 x_1')$$

$$S_1' = a_{11} S_1 + a_{12} S_2 + a_{13} S_3 \tag{2-22a}$$

ıere a_{11} is the direction cosine between x_1' and x_1, a_{12} is the direction cosine

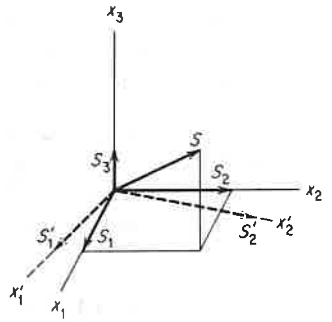


Figure 2-9 Transformation of axes for a vector.

between x_1' and x_2, etc. Similarly,

$$S_2' = a_{21} S_1 + a_{22} S_2 + a_{23} S_3 \tag{2-22b}$$

$$S_3' = a_{31} S_1 + a_{32} S_2 + a_{33} S_3 \tag{2-22c}$$

We note that the leading suffix for each direction cosine in each equation is the same, so we could write these equations as

$$S_1' = \sum_{j=1}^{3} a_{1j} S_j \qquad S_2' = \sum_{j=1}^{3} a_{2j} S_j \qquad S_3' = \sum_{j=1}^{3} a_{3j} S_j$$

These three equations could be combined by writing

$$S_i' = \sum_{j=1}^{3} a_{ij} S_j (i = 1, 2, 3) = a_{i1} S_1 + a_{i2} S_2 + a_{i3} S_3 \tag{2-23}$$

Still greater brevity is obtained by writing Eq. (2-23) in the Einstein suffix notation

$$S_i' = a_{ij} S_j \tag{2-24}$$

The suffix notation is a very useful way of compactly expressing the systems of equations usually found in continuum mechanics. In Eq. (2-24) it is understood that when a suffix occurs twice in the same term (in this case the suffix j), it indicates *summation* with respect to that suffix. Unless otherwise indicated, the summation of the other index is from 1 to 3.

In the above example, i is a *free suffix* and it is understood that in the expanded form there is one equation for each value of i. The repeated index is called a *dummy suffix*. Its only purpose is to indicate summation. Exactly the same three equations would be produced if some other letter were used for the dummy suffix, for example, $S_i' = a_{ir} S_r$ would mean the same thing as Eq. (2-24).

We saw in Sec. 2-5 that the complete determination of the state of stress at a point in a solid requires the specification of nine components of stress on the orthogonal faces of the element at the point. A vector quantity only requires the specification of three components. Obviously, stress is more complicated than a

:ctor. Physical quantities that transform with coordinate axes in the manner of]. (2-18) are called *tensors* of the *second rank*. Stress, strain, and many other ıysical quantities are second-rank tensors. A scalar quantity, which remains ıchanged with transformation of axes, requires only a single number for its ecification. Scalars are tensors of zero rank. Vector quantities require three ımponents for their specification, so they are tensors of the first rank. The ımber of components required to specify a quantity is 3^n, where n is the rank of e tensor.[1] The elastic constant that relates stress with strain in an elastic solid is fourth-rank tensor with 81 components in the general case.

Example The displacements of points in a deformed elastic solid (u) are related to the coordinates of the points (x) by a vector relationship $u_i = e_{ij}x_j$. Expand this tensor expression.

Since j is the dummy suffix, summation will take place over $j = 1, 2, 3$.

$$u_1 = \sum e_{1j}x_j = e_{11}x_1 + e_{12}x_2 + e_{13}x_3$$
$$u_2 = \sum e_{2j}x_j = e_{21}x_1 + e_{22}x_2 + e_{23}x_3$$
$$u_3 = \sum e_{3j}x_j = e_{31}x_1 + e_{32}x_2 + e_{33}x_3$$

The coefficients in these equations are the components of the strain tensor.

The product of two vectors **A** and **B** having components (A_1, A_2, A_3) and $'_1, B_2, B_3)$ results in a second-rank tensor T_{ij}. The components of this tensor n be displayed as a 3×3 matrix.

$$T_{ij} = \begin{vmatrix} T_{11} & T_{12} & T_{13} \\ T_{21} & T_{22} & T_{23} \\ T_{31} & T_{32} & T_{33} \end{vmatrix} = \begin{vmatrix} A_1B_1 & A_1B_2 & A_1B_3 \\ A_2B_1 & A_2B_2 & A_2B_3 \\ A_3B_1 & A_3B_2 & A_3B_3 \end{vmatrix}$$

ı transformation of axes the vector components become (A'_1, A'_2, A'_3) and $'_1, B'_2, B'_3)$. We wish to find the relationship between the nine components of T_{ij} d the nine components of T'_{ij} after the transformation of axes.

$$A'_i = a_{ij}A_j \qquad B'_k = a_{kl}B_l$$
$$A'_iB'_k = (a_{ij}A_j)(a_{kl}B_l) \tag{2-25}$$
$$T'_{ik} = a_{ij}a_{kl}T_{jl}$$

Since stress is a second-rank tensor, the components of the stress tensor can written as

$$\sigma_{ij} = \begin{vmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{vmatrix} = \begin{vmatrix} \sigma_x & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_y & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_z \end{vmatrix}$$

[1] A more precise relationship is $N = k^n$, where N is the number of components required for the cription of a tensor of the nth rank in a k-dimensional space. For a two dimensional space only r components are required to describe a second-rank tensor.

The transformation of the stress tensor σ_{ij} from the x_1, x_2, x_3 system of axes to the x'_1, x'_2, x'_3 axes is given by

$$\sigma_{kl} = a_{ki}a_{lj}\sigma_{ij} \tag{2-26}$$

where i and j are dummy suffixes and k and l are free suffixes. To expand the tensor equation, we first sum over $j = 1, 2, 3$.

$$\sigma_{kl} = a_{ki}a_{l1}\sigma_{i1} + a_{ki}a_{l2}\sigma_{i2} + a_{ki}a_{l3}\sigma_{i3}$$

Now summing over $i = 1, 2, 3$

$$\begin{aligned}\sigma_{kl} = {} & a_{k1}a_{l1}\sigma_{11} + a_{k1}a_{l2}\sigma_{12} + a_{k1}a_{l3}\sigma_{13} \\ & + a_{k2}a_{l1}\sigma_{21} + a_{k2}a_{l2}\sigma_{22} + a_{k2}a_{l3}\sigma_{23} \\ & + a_{k3}a_{l1}\sigma_{31} + a_{k3}a_{l2}\sigma_{32} + a_{k3}a_{l3}\sigma_{33}\end{aligned} \tag{2-27}$$

For each value of k and l there will be an equation similar to (2-27). Thus, to find the equation for the normal stress in the x'_1 direction, let $k = 1$ and $l = 1$

$$\begin{aligned}\sigma_{11} = {} & a_{11}a_{11}\sigma_{11} + a_{11}a_{12}\sigma_{12} + a_{11}a_{13}\sigma_{13} \\ & + a_{12}a_{11}\sigma_{21} + a_{12}a_{12}\sigma_{22} + a_{12}a_{13}\sigma_{23} \\ & + a_{13}a_{11}\sigma_{31} + a_{13}a_{12}\sigma_{32} + a_{13}a_{13}\sigma_{33}\end{aligned}$$

The reader should verify that this reduces to Eq. (2-18) when recast in the symbolism of Sec. 2-5.

Similarly, if we want to determine the shear stress on the x' plane in the z' direction, that is, $\tau_{x'z'}$, let $k = 1$ and $l = 3$

$$\begin{aligned}\sigma_{13} = {} & a_{11}a_{31}\sigma_{11} + a_{11}a_{32}\sigma_{12} + a_{11}a_{33}\sigma_{13} \\ & + a_{12}a_{31}\sigma_{21} + a_{12}a_{32}\sigma_{22} + a_{12}a_{33}\sigma_{23} \\ & + a_{13}a_{31}\sigma_{31} + a_{13}a_{32}\sigma_{32} + a_{13}a_{33}\sigma_{33}\end{aligned}$$

It is perhaps worth emphasizing again that it is immaterial what letters are used for subscripts in tensor notation. Thus, the transformation of a second-rank tensor could just as well be written as $T'_{st} = a_{sp}a_{tq}T_{pq}$, where T_{pq} are the components in the original unprimed axes and T'_{st} are the components referred to the new primed axes. The transformation law for a third-rank tensor is written

$$T'_{stu} = a_{sp}a_{tq}a_{ur}T_{pqr}$$

The material presented so far in this section is really little more than tensor notation. However, even with the minimal topics that have been discussed we have gained a powerful shorthand method for writing the often unwieldy equations of continuum mechanics. (The student will find that this will greatly ease the problem of remembering equations.) We have also gained a useful technique for transforming a tensor quantity from one set of axes to another. There are only a few additional facts about tensors that we need to consider. The student interested in pursuing this topic further is referred to a number of applications-oriented texts on cartesian tensors.[1]

[1] L. G. Jaeger, "Cartesian Tensors in Engineering Science," Pergamon Press, New York, 1966.

A useful quantity in tensor theory is the Kronecker delta δ_{ij}. The Kronecker :lta is a second-rank unit isotropic tensor, that is, it has identical components in ıy coordinate system.

$$\delta_{ij} = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases} \tag{2-28}$$

ultiplication of a tensor or products of tensors by δ_{ij} result in a reduction of 'o in the rank of the tensor. This is called *contraction* of the tensor. The rule is ated here without proof but examples are given so we can make use of this eration in subsequent discussions. Consider the product of two second-rank nsors $A_{pq}B_{vw}$. This multiplication would produce a fourth-rank tensor, nine uations each with nine terms. If we multiply the product by δ_{qw}, it is reduced to second-rank tensor.

$$A_{pq}B_{vw}\delta_{qw} = A_{pq}B_{vq}$$

ıe "rule" is, replace w by q and drop δ_{qw}. The process of contraction can be peated several times. Thus, $A_{pq}B_{vw}\delta_{qw}\delta_{pv}$ reduces to $A_{pq}B_{vq}\delta_{pv}$ on the first ntraction, and then to $A_{pq}B_{pq}$, which is a zero-rank tensor (scalar).

If we apply contraction to the second-rank stress tensor

$$\sigma_{ij}\delta_{ij} = \sigma_{ii} = \sigma_{11} + \sigma_{22} + \sigma_{33} = I_1$$

: obtain the first invariant of the tensor (a scalar).

The invariants of the stress tensor may be determined readily from the matrix its components. Since $\sigma_{12} = \sigma_{21}$, etc., the stress tensor is a *symmetric tensor*.

$$\sigma_{ij} = \begin{vmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{12} & \sigma_{22} & \sigma_{23} \\ \sigma_{13} & \sigma_{23} & \sigma_{33} \end{vmatrix}$$

ıe first invariant is the trace of the matrix, i.e., the sum of the main diagonal ·ms

$$I_1 = \sigma_{11} + \sigma_{22} + \sigma_{33}$$

ıe second invariant is the sum of the principal minors. A minor of an element of natrix is the determinant of the next lower order which remains when the row d column in which the element stands are suppressed. Thus, taking each of the incipal (main diagonal) terms in order and suppressing that row and column we ve

$$I_2 = \begin{vmatrix} \sigma_{22} & \sigma_{23} \\ \sigma_{23} & \sigma_{33} \end{vmatrix} + \begin{vmatrix} \sigma_{11} & \sigma_{13} \\ \sigma_{13} & \sigma_{33} \end{vmatrix} + \begin{vmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{12} & \sigma_{22} \end{vmatrix}$$

nally, the third invariant is the determinant of the entire matrix of the mponents of the stress tensor.

As an example of the advantages of the concepts that are provided by tensor tation we shall derive again the equations for principal stress that were veloped in Sec. 2-5. The reader is warned that it is easy to lose the physical

significance in the mathematical manipulation. It is a basic theorem of tensor theory that there is some orientation of the coordinate axes such that the components of a symmetric tensor of rank 2 will all be equal to zero for $i \neq j$. This is equivalent to stating that the concepts of principal stress and principal axes are inherent in the tensor character of stress.

The three force summation equations, Eqs. (2-17), can be written as

$$\sigma_{nj} = a_{ni}\sigma_{ij} \tag{2-29}$$

where the suffix n is used to denote that we are dealing with the angles to the normal of an oblique plane. If we let the oblique plane be a principal plane and let the normal stress on it be σ_p, then we can write

$$\sigma_{nj} = a_{pj}\sigma_p \tag{2-30}$$

Combining Eqs. (2-29) and (2-30)

$$(a_{ni}\sigma_{ij} - a_{pj}\sigma_p) = 0 \tag{2-31}$$

But, $a_{pj} = a_{pi}\delta_{ji}$ (replace i by j and drop δ_{ji})

$$a_{ni}\sigma_{ij} - \sigma_p a_{pi}\delta_{ji} = 0$$

However, $a_{ni} = a_{pi}$, since the principal stress lies in the direction of the normal to the oblique plane, so

$$(\sigma_{ij} - \sigma_p\delta_{ji})a_{pi} = 0 \tag{2-32}$$

Expanding Eq. (2-32) will give the three equations (2-13), since $a_{p1} = l$, $a_{p2} = m$, etc., and $\delta_{ji} = 0$ when $j \neq i$. For Eq. (2-32) to have a nontrivial solution in a_{pi} the determinant of the coefficients must vanish, resulting in

$$|\sigma_{ij} - \sigma_p\delta_{ji}| = \begin{vmatrix} \sigma_x - \sigma_p & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_y - \sigma_p & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_z - \sigma_p \end{vmatrix} = 0$$

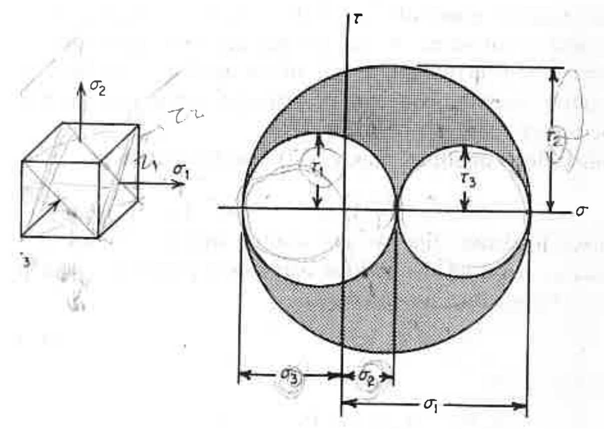
which yields the cubic equation Eq. (2-14). The coefficients of this equation in tensor notation are

$$I_1 = \sigma_{ii}$$
$$I_2 = \tfrac{1}{2}(\sigma_{ik}\sigma_{ki} - \sigma_{ii}\sigma_{kk})$$
$$I_3 = \tfrac{1}{6}(2\sigma_{ij}\sigma_{jk}\sigma_{ki} - 3\sigma_{ij}\sigma_{ji}\sigma_{kk} + \sigma_{ii}\sigma_{jj}\sigma_{kk})$$

The fact that only dummy subscripts appear in these equations indicates the scalar nature of the invariants of the stress tensor.

2-7 MOHR'S CIRCLE—THREE DIMENSIONS

The discussion given in Sec. 2-4 of the representation of a two-dimensional state of stress by means of Mohr's circle can be extended to three dimensions. Figure 2-10 shows how a triaxial state of stress, defined by the three principal stresses,



ure 2-10 Mohr's circle representation of a three-dimensional state of stress.

ı be represented by three Mohr's circles. It can be shown[1] that all possible ess conditions within the body fall within the shaded area between the circles in ;. 2-10.

While the only physical significance of Mohr's circle is that it gives a ›metrical representation of the equations that express the transformation of ess components to different sets of axes, it is a very convenient way visualizing the state of stress. Figure 2-11 shows Mohr's circle for a number of nmon states of stress. Note that the application of a tensile stress σ_2 at right gles to an existing tensile stress σ_1 (Fig. 2-11*c*) results in a decrease in the ncipal shear stress on two of the three sets of planes on which a principal shear ess acts. However, the maximum shear stress is not decreased from what it uld be for uniaxial tension, although if only the two-dimensional Mohr's circle l been used, this would not have been apparent. If a tensile stress is applied in third principal direction (Fig. 2-11*d*), the maximum shear stress is reduced ›reciably. For the limiting case of equal triaxial tension (hydrostatic tension) ›hr's circle reduces to a point, and there are no shear stresses acting on any ne in the body. The effectiveness of biaxial- and triaxial-tension stresses in ucing the shear stresses results in a considerable decrease in the ductility of the terial, because plastic deformation is produced by shear stresses. Thus, brittle cture is invariably associated with triaxial stresses developed at a notch or ss raiser. However, Fig. 2-11*e* shows that, if compressive stresses are applied ral to a tensile stress, the maximum shear stress is larger than for the case of ıer uniaxial tension or compression. Because of the high value of shear stress ıtive to the applied tensile stress the material has an excellent opportunity to

[1] A. Nadai, "Theory of Flow and Fracture of Solids," 2d ed., pp. 96–98, McGraw-Hill Book ıpany, New York, 1950.

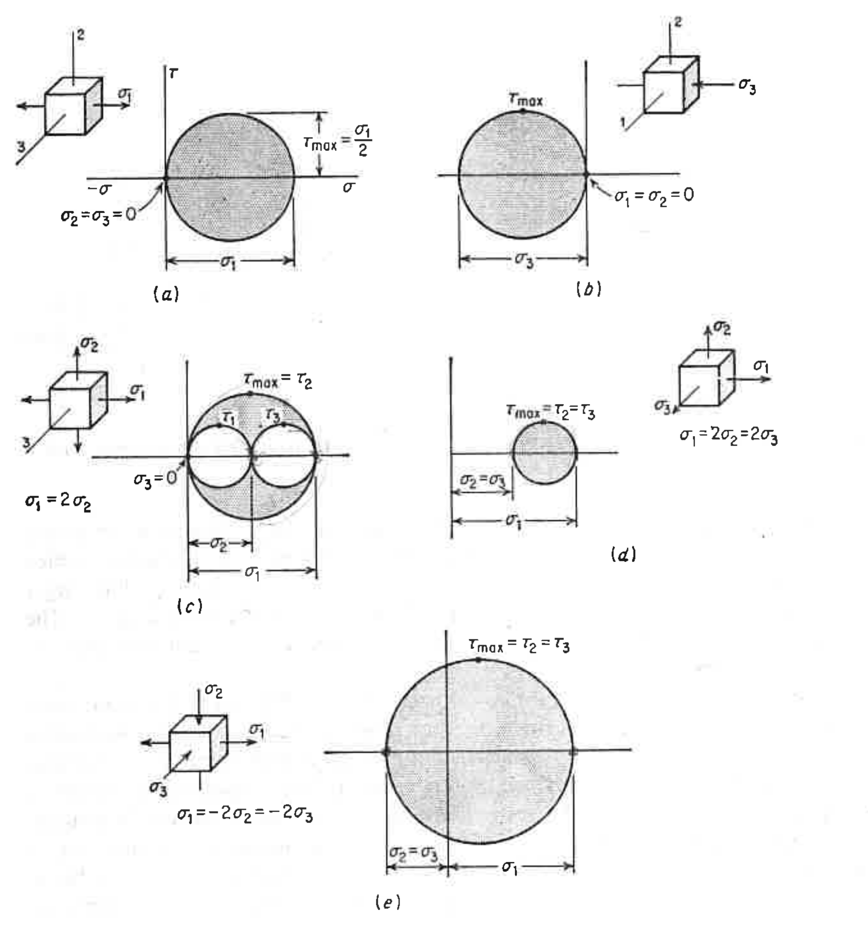
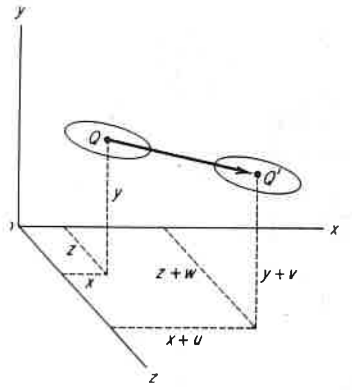


Figure 2-11 Mohr's circles (three-dimensional) for various states of stress. (*a*) Uniaxial tension; (*b*) uniaxial compression; (*c*) biaxial tension; (*d*) triaxial tension (unequal); (*e*) uniaxial tension plus biaxial compression.

deform plastically without fracturing under this state of stress. Important use is made of this fact in the plastic working of metals. For example, greater ductility is obtained in extrusion through a die than in simple uniaxial tension because the reaction of the metal with the die will produce lateral compressive stresses.

2-8 DESCRIPTION OF STRAIN AT A POINT

The displacement of points in a continuum may result from rigid-body translation, rotation, and deformation. The deformation of a solid may be made up of



igure 2-12 Displacement of point Q.

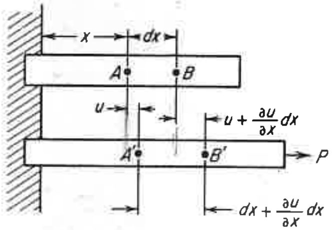


Fig. 2-13 One-dimensional strain.

ilatation, change in volume, or *distortion*, change in shape. Situations involving ·anslation and rotation are usually treated in the branch of mechanics called *ynamics*. Small deformations are the province of elasticity theory, while larger eformations are treated in the disciplines of plasticity and hydrodynamics. The quations developed in this section are basically geometrical, so that they apply to ll types of continuous media.

Consider a solid body in fixed coordinates, x, y, z (Fig. 2-12). Let a combina-on of deformation and movement displace point Q to Q' with new coordinates $+ u$, $y + v$, $z + w$. The components of the displacement are u, v, w. The dis-lacement of Q is the vector $\mathbf{u}_Q = f(u, v, w)$. If the displacement vector is onstant for all particles in the body then there is no strain. However, in general, $_i$ is different from particle to particle so that displacement is a function of istance, $u_i = f(x_i)$. For elastic solids and small displacements, u_i is a linear ınction of x_i, homogeneous displacements, and the displacement equations are near. However, for other materials the displacement may not be linear with istance, which leads to cumbersome mathematical relationships.

To start our discussion of strain, consider a simple one-dimensional case (Fig. -13). In the undeformed state points A and B are separated by a distance dx. /hen a force is applied in the x direction A moves to A' and B moves to B'. ince displacement u, in this one-dimensional case, is a function of x, B is isplaced slightly more than A since it is further from the fixed end. The normal rain is given by

$$e_x = \frac{\Delta L}{L} = \frac{A'B' - AB}{AB} = \frac{dx + \dfrac{\partial u}{\partial x}dx - dx}{dx} = \frac{\partial u}{\partial x} \qquad (2\text{-}33)$$

For this one-dimensional case, the displacement is given by $u = e_x x$. To eneralize this to three dimensions, each of the components of the displacement

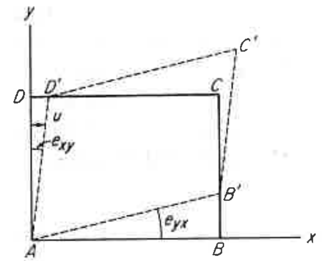


Figure 2-14 Angular distortion of an element.

will be linearly related to each of the three initial coordinates of the point.

$$\begin{aligned} u &= e_{xx}x + e_{xy}y + e_{xz}z \\ v &= e_{yx}x + e_{yy}y + e_{yz}z \\ w &= e_{zx}x + e_{zy}y + e_{zz}z \end{aligned} \qquad (2\text{-}34)$$

or

$$u_i = e_{ij}x_j \qquad (2\text{-}35)$$

The coefficients relating displacement with the coordinates of the point in the body are the components of the relative displacement tensor. Three of these terms can be identified readily as the normal strains.

$$e_{xx} = \frac{\partial u}{\partial x} \qquad e_{yy} = \frac{\partial v}{\partial y} \qquad e_{zz} = \frac{\partial w}{\partial z} \qquad (2\text{-}36)$$

However, the other six coefficients require further scrutiny.

Consider an element in the xy plane which has been distorted by shearing stresses (Fig. 2-14). The element has undergone angular distortion. The displacement of points along the line AD is parallel to the x axis, but this component of displacement increases in proportion to the distance out along the y axis. Thus, referring to Eq. (2-34)

$$e_{xy} = \frac{DD'}{DA} = \frac{\partial u}{\partial y} \qquad (2\text{-}37)$$

Similarly, for the angular distortion of the x axis

$$e_{yx} = \frac{BB'}{AB} = \frac{\partial v}{\partial x} \qquad (2\text{-}38)$$

These shear displacements are positive when they rotate a line from one positive axis towards another positive axis. By similar methods the rest of the components of the displacement tensor can be seen to be

$$e_{ij} = \begin{vmatrix} e_{xx} & e_{xy} & e_{xz} \\ e_{yx} & e_{yy} & e_{yz} \\ e_{zx} & e_{zy} & e_{zz} \end{vmatrix} = \begin{vmatrix} \dfrac{\partial u}{\partial x} & \dfrac{\partial u}{\partial y} & \dfrac{\partial u}{\partial z} \\ \dfrac{\partial v}{\partial x} & \dfrac{\partial v}{\partial y} & \dfrac{\partial v}{\partial z} \\ \dfrac{\partial w}{\partial x} & \dfrac{\partial w}{\partial y} & \dfrac{\partial w}{\partial z} \end{vmatrix} \qquad (2\text{-}39)$$

$$\frac{A'B' - AB}{AB} = \frac{dx + \frac{du}{dx}dx - dx}{dx} = \frac{du}{dx}$$

In general, displacement components such as e_{xy}, e_{yx}, etc., produce both *shear strain* and *rigid-body rotation*. Figure 2-15 illustrates several cases. Since we need to identify that part of the displacement that results in strain, it is important to break the displacement tensor into a strain contribution and a rotational contribution. Fortunately, a basic postulate of tensor theory states that any second-rank tensor can be decomposed into a symmetric tensor and an antisymmetric (skew-symmetric) tensor.

$$e_{ij} = \tfrac{1}{2}(e_{ij} + e_{ji}) + \tfrac{1}{2}(e_{ij} - e_{ji}) \tag{2-40}$$

or

$$e_{ij} = \varepsilon_{ij} + \omega_{ij} \tag{2-41}$$

where $\varepsilon_{ij} = \dfrac{1}{2}\left(\dfrac{\partial u_i}{\partial x_j} + \dfrac{\partial u_j}{\partial x_i}\right)$ and is called the *strain tensor*

$\omega_{ij} = \dfrac{1}{2}\left(\dfrac{\partial u_i}{\partial x_j} - \dfrac{\partial u_j}{\partial x_i}\right)$ and is called the *rotation tensor*

$$\varepsilon_{ij} = \begin{vmatrix} \varepsilon_{xx} & \varepsilon_{xy} & \varepsilon_{xz} \\ \varepsilon_{yx} & \varepsilon_{yy} & \varepsilon_{yz} \\ \varepsilon_{zx} & \varepsilon_{zy} & \varepsilon_{zz} \end{vmatrix} = \begin{vmatrix} \dfrac{\partial u}{\partial x} & \dfrac{1}{2}\left(\dfrac{\partial u}{\partial y} + \dfrac{\partial v}{\partial x}\right) & \dfrac{1}{2}\left(\dfrac{\partial u}{\partial z} + \dfrac{\partial w}{\partial x}\right) \\ \dfrac{1}{2}\left(\dfrac{\partial u}{\partial y} + \dfrac{\partial v}{\partial x}\right) & \dfrac{\partial v}{\partial y} & \dfrac{1}{2}\left(\dfrac{\partial v}{\partial z} + \dfrac{\partial w}{\partial y}\right) \\ \dfrac{1}{2}\left(\dfrac{\partial u}{\partial z} + \dfrac{\partial w}{\partial x}\right) & \dfrac{1}{2}\left(\dfrac{\partial v}{\partial z} + \dfrac{\partial w}{\partial y}\right) & \dfrac{\partial w}{\partial z} \end{vmatrix} \tag{2-42}$$

$$\omega_{ij} = \begin{vmatrix} \omega_{xx} & \omega_{xy} & \omega_{xz} \\ \omega_{yx} & \omega_{yy} & \omega_{yz} \\ \omega_{zx} & \omega_{zy} & \omega_{zz} \end{vmatrix} = \begin{vmatrix} 0 & \dfrac{1}{2}\left(\dfrac{\partial u}{\partial y} - \dfrac{\partial v}{\partial x}\right) & \dfrac{1}{2}\left(\dfrac{\partial u}{\partial z} - \dfrac{\partial w}{\partial x}\right) \\ \dfrac{1}{2}\left(\dfrac{\partial v}{\partial x} - \dfrac{\partial u}{\partial y}\right) & 0 & \dfrac{1}{2}\left(\dfrac{\partial v}{\partial z} - \dfrac{\partial w}{\partial y}\right) \\ \dfrac{1}{2}\left(\dfrac{\partial w}{\partial x} - \dfrac{\partial u}{\partial z}\right) & \dfrac{1}{2}\left(\dfrac{\partial w}{\partial y} - \dfrac{\partial v}{\partial z}\right) & 0 \end{vmatrix} \tag{2-43}$$

Note that ε_{ij} is a symmetric tensor since $\varepsilon_{ij} = \varepsilon_{ji}$, that is, $\varepsilon_{xy} = \varepsilon_{xz}$, etc. ω_{ij} is an antisymmetric tensor since $\omega_{ij} = -\omega_{ji}$, that is, $\omega_{xy} = -\omega_{yx}$. If $\omega_{ij} = 0$, the deformation is said to be irrotational.

By substituting Eq. (2-41) into Eq. (2-35), we get the general displacement equations

$$u_i = \varepsilon_{ij}x_j + \omega_{ij}x_j \tag{2-44}$$

Earlier in Sec. 1-9 the shear strain γ was defined as the total angular change from a right angle. Referring to Fig. 2-15*a*, $\gamma = e_{xy} + e_{yx} = \varepsilon_{xy} + \varepsilon_{yx} = 2\varepsilon_{xy}$.

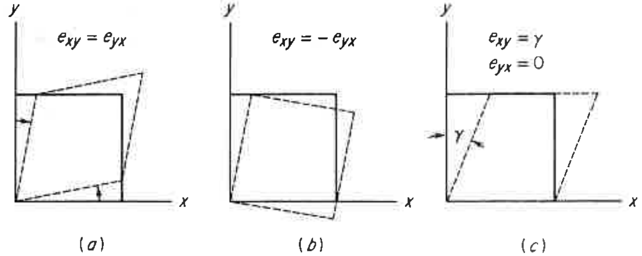


Figure 2-15 Some examples of displacement with shear and rotation. (*a*) Pure shear without rotation; (*b*) pure rotation without shear; (*c*) simple shear. Simple shear involves a shape change produced by displacements along a single set of parallel planes. Pure shear involves a shape change produced by equal shear displacements on two sets of perpendicular planes.

This definition of shear strain, $\gamma_{ij} = 2\varepsilon_{ij}$, is called the *engineering shear strain*.

$$\begin{aligned} \gamma_{xy} &= \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \\ \gamma_{xz} &= \frac{\partial w}{\partial x} + \frac{\partial u}{\partial z} \\ \gamma_{yz} &= \frac{\partial w}{\partial y} + \frac{\partial v}{\partial z} \end{aligned} \tag{2-45}$$

This definition of shear strain commonly is used in engineering elasticity. However, the shear strain defined according to Eq. (2-45) *is not a tensor quantity*.

Because of the obvious advantages in the transformation of tensors by the methods discussed in Sec. 2-6, it is profitable to use the strain tensor as defined by Eq. (2-42). Since the strain tensor is a second-rank tensor, it has all of the properties that have been described earlier for stress. Thus, the strain tensor may be transformed from one set of coordinate axes to a new system of axes by

$$\varepsilon_{kl} = a_{ki}a_{lj}\varepsilon_{ij} \tag{2-46}$$

For simplicity, equations for strain analogous with those for stress can be written directly by substituting ε for σ and $\gamma/2$ for τ. Thus, the normal strain on an oblique plane is given by

$$\varepsilon = \varepsilon_x l^2 + \varepsilon_y m^2 + \varepsilon_z n^2 + \gamma_{xy} lm + \gamma_{yz} mn + \gamma_{xz} ln$$

[Compare the above with Eq. (2-18).]

In complete analogy with stress, it is possible to define a system of coordinate axes along which there are no shear strains. These axes are the principal strain axes. For an isotropic body the direction of principal strains coincide with principal stress directions.[1] An element oriented along one of the principal strain axes will undergo pure extension or contraction without any rotation or shear strain. The three principal strains are the roots of the cubic equation

$$\varepsilon^3 - I_1\varepsilon^2 + I_2\varepsilon - I_3 = 0 \tag{2-47}$$

where

$$I_1 = \varepsilon_x + \varepsilon_y + \varepsilon_z$$

$$I_2 = \varepsilon_x\varepsilon_y + \varepsilon_y\varepsilon_z + \varepsilon_z\varepsilon_x - \tfrac{1}{4}\left(\gamma_{xy}^2 + \gamma_{zx}^2 + \gamma_{yz}^2\right)$$

$$I_3 = \varepsilon_x\varepsilon_y\varepsilon_z + \tfrac{1}{4}\gamma_{yx}\gamma_{zx}\gamma_{yz} - \tfrac{1}{4}\left(\varepsilon_x\gamma_{yz}^2 + \varepsilon_y\gamma_{zx}^2 + \varepsilon_z\gamma_{xy}^2\right)$$

The directions of the principal strains are obtained from the three equations analogous to Eqs. (2-13)

$$2l(\varepsilon_x - \varepsilon) + m\gamma_{xy} + n\gamma_{xz} = 0$$

$$l\gamma_{xy} + 2m(\varepsilon_y - \varepsilon) + n\gamma_{yz} = 0$$

$$l\gamma_{xz} + m\gamma_{yz} + 2n(\varepsilon_z - \varepsilon) = 0$$

Continuing the analogy between stress and strain equations, the equation for the *principal shearing strains* can be obtained from Eq. (2-20).

$$\gamma_1 = \varepsilon_2 - \varepsilon_3$$

$$\gamma_{\max} = \gamma_2 = \varepsilon_1 - \varepsilon_3 \tag{2-48}$$

$$\gamma_3 = \varepsilon_1 - \varepsilon_2$$

In general, the deformation of a solid involves a combination of volume change and change in shape. Therefore, we need a way to determine how much of the deformation is due to these contributions. The *volume strain*, or cubical dilatation, is the change in volume per unit volume. Consider a rectangular parallelepiped with edges dx, dy, dz. The volume in the strained condition is $(1 + \varepsilon_x)(1 + \varepsilon_y)(1 + \varepsilon_z)\,dx\,dy\,dz$, since only normal strains result in volume change. The volume strain Δ is

$$\Delta = \frac{(1 + \varepsilon_x)(1 + \varepsilon_y)(1 + \varepsilon_z)\,dx\,dy\,dz - dx\,dy\,dz}{dx\,dy\,dz}$$

$$= (1 + \varepsilon_x)(1 + \varepsilon_y)(1 + \varepsilon_z) - 1$$

which for small strains, after neglecting the products of strains, becomes

$$\Delta = \varepsilon_x + \varepsilon_y + \varepsilon_z \tag{2-49}$$

Note that the volume strain is equal to the first invariant of the strain tensor, $\Delta = \varepsilon_x + \varepsilon_y + \varepsilon_z = \varepsilon_1 + \varepsilon_2 + \varepsilon_2$. We can also define $(\varepsilon_x + \varepsilon_y + \varepsilon_z)/3$ as the *mean strain* or the hydrostatic (spherical) component of strain.

$$\varepsilon_m = \frac{\varepsilon_x + \varepsilon_y + \varepsilon_z}{3} = \frac{\varepsilon_{kk}}{3} = \frac{\Delta}{3} \tag{2-50}$$

[1] For a derivation of this point see C. T. Wang, "Applied Elasticity," pp. 26–27, McGraw-Hill Book Company, New York, 1953.

That part of the strain tensor which is involved in shape change rather than volume change is called the *strain deviator* ε'_{ij}. To obtain the deviatoric strains, we simply subtract ε_m from each of the normal strain components. Thus,

$$\varepsilon'_{ij} = \begin{vmatrix} \varepsilon_x - \varepsilon_m & \varepsilon_{xy} & \varepsilon_{xz} \\ \varepsilon_{yx} & \varepsilon_y - \varepsilon_m & \varepsilon_{yz} \\ \varepsilon_{zx} & \varepsilon_{zy} & \varepsilon_z - \varepsilon_m \end{vmatrix}$$

$$= \begin{vmatrix} \dfrac{2\varepsilon_x - \varepsilon_y - \varepsilon_z}{3} & \varepsilon_{xy} & \varepsilon_{xz} \\ \varepsilon_{yx} & \dfrac{2\varepsilon_y - \varepsilon_z - \varepsilon_x}{3} & \varepsilon_{yz} \\ \varepsilon_{zx} & \varepsilon_{zy} & \dfrac{2\varepsilon_z - \varepsilon_x - \varepsilon_y}{3} \end{vmatrix} \tag{2-51}$$

The division of the total strain tensor into deviatoric and dilatational strains is given in tensor notation by

$$\varepsilon_{ij} = \varepsilon'_{ij} + \varepsilon_m = \left(\varepsilon_{ij} - \frac{\Delta}{3}\delta_{ij}\right) + \frac{\Delta}{3}\delta_{ij} \tag{2-52}$$

For example, when ε_{ij} are the principal strains, $(i = j)$, the strain deviators are $\varepsilon'_{11} = \varepsilon_{11} - \varepsilon_m$, $\varepsilon'_{22} = \varepsilon_{22} - \varepsilon_m$, $\varepsilon'_{33} = \varepsilon_{33} - \varepsilon_m$. These strains represent elongations or contractions along the principal axes that change the shape of the body at constant volume.

2-9 MOHR'S CIRCLE OF STRAIN

Except in a few cases involving contact stresses, it is not possible to measure stress directly. Therefore, experimental measurements of stress are actually based on measured strains and are converted to stresses by means of Hooke's law and the more general relationships which are given in Sec. 2-11. The most universal strain-measuring device is the bonded-wire resistance gage, frequently called the SR-4 strain gage.[1] These gages are made up of several loops of fine wire or foil of special composition, which are bonded to the surface of the body to be studied. When the body is deformed, the wires in the gage are strained and their electrical resistance is altered. The change in resistance, which is proportional to strain, can be accurately determined with a simple Wheatstone-bridge circuit. The high sensitivity, stability, comparative ruggedness, and ease of application make resistance strain gages a very powerful tool for strain determination.

[1] For a treatment of strain gages and other techniques of experimental stress analysis see J. W. Dally, and W. F. Riley, "Experimental Stress Analysis," 2d ed., McGraw-Hill Book Company, New York, 1978.

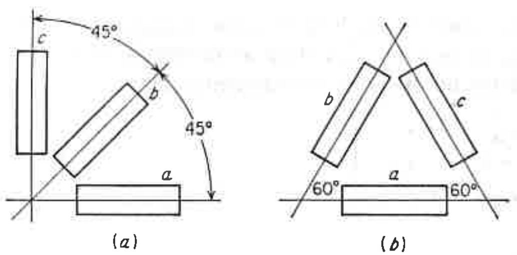


Figure 2-16 Typical strain-gage rosettes. (*a*) Rectangular; (*b*) delta.

For practical problems of experimental stress analysis if is often important to determine the principal stresses. If the principal directions are known, gages can be oriented in these directions and the principal stresses determined quite readily. In the general case the direction of the principal strains will not be known, so that it will be necessary to determine the orientation and magnitude of the principal strains from the measured strains in arbitrary directions. Because no stress can act perpendicular to a free surface, strain-gage measurements involve a two-dimensional state of stress. The state of strain is completely determined if ε_x, ε_y, and γ_{xy} can be measured. However, strain gages can make only direct readings of linear strain, while shear strains must be determined indirectly. Therefore, it is the usual practice to use three strain gages separated at fixed angles in the form of a "rosette," as in Fig. 2-16. Strain-gage readings at three values of θ will give three simultaneous equations similar to Eq. (2-53) which can be solved for ε_x, ε_y, and γ_{xy}. The two-dimensional version of Eq. (2-47) can then be used to determine the principal strains.

$$\varepsilon_\theta = \varepsilon_x \cos^2\theta + \varepsilon_y \sin^2\theta + \gamma_{xy}\sin\theta\cos\theta \qquad (2\text{-}53)$$

A more convenient method of determining the principal strains from strain-gage readings than the solution of three simultaneous equations in three unknowns is the use of Mohr's circle. In constructing a Mohr's circle representation of strain, values of linear normal strain ε are plotted along the x axis, and the shear strain divided by 2 is plotted along the y axis. Figure 2-17 shows the Mohr's circle construction[1] for the generalized strain-gage rosette illustrated at the top of the figure. Strain-gage readings ε_a, ε_b, and ε_c are available for three gages situated at arbitrary angles α and β. The objective is to determine the magnitude and orientation of the principal strains ε_1 and ε_2.

1. Along an arbitrary axis $X'X'$ lay off vertical lines aa, bb, and cc corresponding to the strains ε_a, ε_b, and ε_c.
2. From any point on the line bb (middle strain gage) draw a line DA at an angle α with bb and intersecting aa at point A. In the same way, lay off DC intersecting cc at point C.

[1] G. Murphy, *J. Appl. Mech.*, vol. 12, p. A209, 1945; F. A. McClintock, *Proc. Soc. Exp. Stress Anal.*, vol. 9, p. 209, 1951.

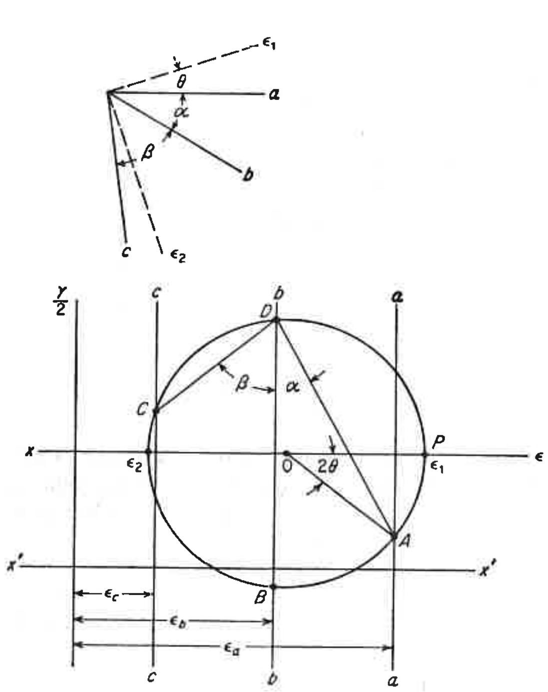


Figure 2-17 Mohr's circle for determination of principal strains.

3. Construct a circle through A, C, and D. The center of this circle is at O, determined by the intersection of the perpendicular bisectors to CD and AD.
4. Points A, B, and C on the circle give the values of ε and $\gamma/2$ (measured from the new x axis through O) for the three gages.
5. Values of the principal strains are determined by the intersection of the circle with the new x axis through O. The angular relationship of ε_1 to the gage a is one-half the angle AOP on the Mohr's circle ($AOP = 2\theta$).

2-10 HYDROSTATIC AND DEVIATOR COMPONENTS OF STRESS

Having introduced the concept that the strain tensor can be divided into a hydrostatic or mean strain and a strain deviator, it is important to consider the physical significance of a similar operation on the stress tensor. The total stress tensor can be divided into a *hydrostatic* or *mean stress tensor* σ_m, which involves only pure tension or compression, and a *deviator stress tensor* σ'_{ij}, which represents the shear stresses in the total state of stress (Fig. 2-18). In direct analogy with the situation for strain, the hydrostatic component of the stress tensor produces only elastic volume changes and does not cause plastic deformation. Experiment shows that the yield stress of metals is independent of hydrostatic

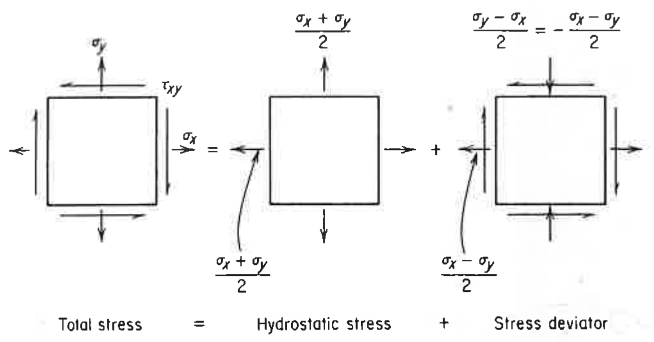


Figure 2-18 Resolution of total stress into hydrostatic stress and stress deviator.

stress, although the fracture strain is strongly influenced by hydrostatic stress. Because the stress deviator involves the shearing stresses, it is important in causing plastic deformation. In Chap. 3 we shall see that the stress deviator is useful in formulating theories of yielding.

The hydrostatic or mean stress is given by

$$\sigma_m = \frac{\sigma_{kk}}{3} = \frac{\sigma_x + \sigma_y + \sigma_z}{3} = \frac{\sigma_1 + \sigma_2 + \sigma_3}{3} \qquad (2\text{-}54)$$

The decomposition of the stress tensor is given by

$$\sigma_{ij} = \sigma'_{ij} + \tfrac{1}{3}\delta_{ij}\sigma_{kk} \qquad (2\text{-}55)$$

Therefore,

$$\sigma'_{ij} = \sigma_{ij} - \sigma_m\delta_{ij} \qquad (2\text{-}56)$$

$$\sigma'_{ij} = \begin{vmatrix} \dfrac{2\sigma_x - \sigma_y - \sigma_z}{3} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \dfrac{2\sigma_y - \sigma_z - \sigma_x}{3} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \dfrac{2\sigma_z - \sigma_x - \sigma_y}{3} \end{vmatrix} \qquad (2\text{-}57)$$

It can be seen readily that the stress deviator involves shear stresses. For example, referring σ'_{ij} to a system of principal axes,

$$\sigma'_1 = \frac{2\sigma_1 - \sigma_2 - \sigma_3}{3} = \frac{(\sigma_1 - \sigma_2) + (\sigma_1 - \sigma_3)}{3}$$

$$\sigma'_1 = \frac{2}{3}\left(\frac{\sigma_1 - \sigma_2}{2} + \frac{\sigma_1 - \sigma_3}{2}\right) = \frac{2}{3}(\tau_3 + \tau_2) \qquad (2\text{-}58)$$

where τ_3 and τ_2 are principal shearing stresses.

Since σ'_{ij} is a second-rank tensor, it has principal axes. The principal values of the stress deviator are the roots of the cubic equation[1]

$$(\sigma')^3 - J_1(\sigma')^2 - J_2\sigma' - J_3 = 0 \qquad (2\text{-}59)$$

where J_1, J_2, J_3 are the invariants of the deviator stress tensor. J_1 is the sum of the principal terms in the diagonal of the matrix of components of σ'_{ij}.

$$J_1 = (\sigma_x - \sigma_m) + (\sigma_y - \sigma_m) + (\sigma_z - \sigma_m) = 0 \qquad (2\text{-}60)$$

J_2 can be obtained from the sum of the principal minors of σ'_{ij}.

$$J_2 = \tau_{xy}^2 + \tau_{yz}^2 + \tau_{xz}^2 - \sigma'_x\sigma'_y - \sigma'_y\sigma'_z - \sigma'_z\sigma'_x$$

$$= \tfrac{1}{6}\left[(\sigma_x - \sigma_y)^2 + (\sigma_y - \sigma_z)^2 + (\sigma_z - \sigma_x)^2 + 6\left(\tau_{xy}^2 + \tau_{yz}^2 + \tau_{xz}^2\right)\right] \qquad (2\text{-}61)$$

The third invariant J_3 is the determinant of Eq. (2-57).

2-11 ELASTIC STRESS-STRAIN RELATIONS

Up till now our discussion of stress and strain has been perfectly general and applicable to any continuum. Now, if we want to relate the stress tensor with the strain tensor, we must introduce the properties of the material. Equations of this nature are called *constitutive equations*. In this chapter we shall consider only constitutive equations for elastic solids. Moreover, initially we shall only consider isotropic elastic solids.

In Chap. 1 we saw that elastic stress is linearly related to elastic strain by means of the modulus of elasticity (Hooke's law).

$$\sigma_x = E\varepsilon_x \qquad (2\text{-}62)$$

where E is the modulus of elasticity in tension or compression. While a tensile force in the x direction produces an extension along that axis, it also produces a contraction in the transverse y and z directions. The transverse strain has been found by experience to be a constant fraction of the strain in the longitudinal direction. This is known as *Poisson's ratio*, denoted by the symbol ν.

$$\varepsilon_y = \varepsilon_z = -\nu\varepsilon_x = -\frac{\nu\sigma_x}{E} \qquad (2\text{-}63)$$

Only the absolute value of ν is used in calculations. For most metals the values[2] of ν are close to 0.33.

To develop the stress-strain relations for a three-dimensional state of stress, consider a unit cube subjected to normal stresses σ_x, σ_y, σ_z and shearing stresses τ_{xy}, τ_{yz}, τ_{zx}. Because the elastic stresses are small and the material is isotropic, we can assume that normal stress σ_x does not produce shear strain on the x, y, or z planes and that a shear stress τ_{xy} does not produce normal strains on the x, y, or

[1] Note that we use a negative sign for the coefficient of σ'. Compare with Eq. (2-14).

[2] W. Koster and H. Franz, *Metall. Rev.*, vol. 6, pp. 1–55, 1961.

z planes. We can then apply the principle of superposition[1] to determine the strain produced by more than one stress component. For example, the stress σ_x produces a normal strain ε_x and two transverse strains $\varepsilon_y = -\nu\varepsilon_x$ and $\varepsilon_z = -\nu\varepsilon_x$. Thus,

Stress	Strain in the x direction	Strain in the y direction	Strain in the z direction
σ_x	$\varepsilon_x = \frac{\sigma_x}{E}$	$\varepsilon_y = -\frac{\nu\sigma_x}{E}$	$\varepsilon_z = -\frac{\nu\sigma_x}{E}$
σ_y	$\varepsilon_x = -\frac{\nu\sigma_y}{E}$	$\varepsilon_y = \frac{\sigma_y}{E}$	$\varepsilon_z = -\frac{\nu\sigma_y}{E}$
σ_z	$\varepsilon_x = -\frac{\nu\sigma_z}{E}$	$\varepsilon_y = -\frac{\nu\sigma_z}{E}$	$\varepsilon_z = \frac{\sigma_z}{E}$

By superposition of the components of strain in the x, y, and z directions

$$\begin{aligned} \varepsilon_x &= \frac{1}{E}\left[\sigma_x - \nu(\sigma_y + \sigma_z)\right] \\ \varepsilon_y &= \frac{1}{E}\left[\sigma_y - \nu(\sigma_z + \sigma_x)\right] \\ \varepsilon_z &= \frac{1}{E}\left[\sigma_z - \nu(\sigma_x + \sigma_y)\right] \end{aligned} \tag{2-64}$$

The shearing stresses acting on the unit cube produce shearing strains.

$$\tau_{xy} = G\gamma_{xy} \qquad \tau_{yz} = G\gamma_{yz} \qquad \tau_{xz} = G\gamma_{xz} \tag{2-65}$$

The proportionality constant G is the *modulus of elasticity in shear*, or the *modulus of rigidity*. Values of G are usually determined from a torsion test.

We have seen that the stress-strain equations for an *isotropic* elastic solid involve three constants, E, G, and ν. Typical values of these constants for a number of metals are given in Table 2-1.

Still another elastic constant is the *bulk modulus* or the *volumetric modulus of elasticity* K. The bulk modulus is the ratio of the hydrostatic pressure to the dilatation that it produces

$$K = \frac{\sigma_m}{\Delta} = \frac{-p}{\Delta} = \frac{1}{\beta} \tag{2-66}$$

where $-p$ is the hydrostatic pressure and β is the compressibility.

Many useful relationships may be derived between the elastic constants E, G, ν, K. For example, if we add up the three equations (2-64),

$$\varepsilon_x + \varepsilon_y + \varepsilon_z = \frac{1-2\nu}{E}(\sigma_x + \sigma_y + \sigma_z)$$

[1] The principle of superposition states that two strains may be combined by direct superposition. The order of application has no effect on the final strain of the body.

Table 2-1 Typical room-temperature values of elastic constants for isotropic materials

Material	Modulus of elasticity, GPa	Shear modulus, GPa	Poisson's ratio
Aluminum alloys	72.4	27.5	0.31
Copper	110	41.4	0.33
Steel (plain carbon and low-alloy)	200	75.8	0.33
Stainless steel (18-8)	193	65.6	0.28
Titanium	117	44.8	0.31
Tungsten	400	157	0.27

The term on the left is the volume strain Δ, and the term on the right is $3\sigma_m$.

$$\Delta = \frac{1-2\nu}{E}3\sigma_m$$

or

$$K = \frac{\sigma_m}{\Delta} = \frac{E}{3(1-2\nu)} \tag{2-67}$$

Another important relationship is the expression relating E, G, and ν. This equation is usually developed in a first course in strength of materials.[1]

$$G = \frac{E}{2(1+\nu)} \tag{2-68}$$

Many other relationships can be developed between these four isotropic elastic constants. For example,

$$E = \frac{9K}{1 + 3K/G} \qquad \nu = \frac{1 - 2G/3K}{2 + 2G/3K}$$

$$G = \frac{3(1-2\nu)K}{2(1+\nu)} \qquad K = \frac{E}{9 - 3E/G}$$

Equations (2-64) and (2-65) may be expressed succinctly in tensor notation

$$\varepsilon_{ij} = \frac{1+\nu}{E}\sigma_{ij} - \frac{\nu}{E}\sigma_{kk}\delta_{ij} \tag{2-69}$$

[1] For a geometric development see D. C. Drucker, "Introduction to Mechanics of Deformable Solids," pp. 64–65, McGraw-Hill Book Company, New York, 1967. For a derivation based on isotropy and transformation of axes see Chou and Pagano, op. cit., pp. 58–59.

For example, if $i = j = x$,

$$\varepsilon_{xx} = \frac{1+\nu}{E}\sigma_{xx} - \frac{\nu}{E}(\sigma_{xx} + \sigma_{yy} + \sigma_{zz})(1)$$
$$= \frac{1}{E}\left[\sigma_{xx} - \nu(\sigma_{yy} + \sigma_{zz})\right]$$

If $i = x$ and $j = y$,

$$\varepsilon_{xy} = \frac{\gamma_{xy}}{2} = \frac{1+\nu}{E}\tau_{xy} - \frac{\nu}{E}\sigma_{kk}(0)$$

where

$$\frac{1+\nu}{E} = \frac{1}{2G} \quad \text{and} \quad \gamma_{xy} = \frac{1}{G}\tau_{xy}$$

2-12 CALCULATION OF STRESSES FROM ELASTIC STRAINS

Since for small elastic strains there is no coupling between the expressions for normal stress and strain and the equations for shear stress and shear strain, it is possible to invert Eqs. (2-64) and (2-65) to solve for stress in terms of strain. From Eq. (2-64),

$$\sigma_x + \sigma_y + \sigma_z = \frac{E}{1-2\nu}(\varepsilon_x + \varepsilon_y + \varepsilon_z) \tag{2-70}$$

$$\varepsilon_x = \frac{1+\nu}{E}\sigma_x - \frac{\nu}{E}(\sigma_x + \sigma_y + \sigma_z) \tag{2-71}$$

Substitution of Eq. (2-70) into Eq. (2-71) gives

$$\sigma_x = \frac{E}{1+\nu}\varepsilon_x + \frac{\nu E}{(1+\nu)(1-2\nu)}(\varepsilon_x + \varepsilon_y + \varepsilon_z) \tag{2-72}$$

or in tensor notation

$$\sigma_{ij} = \frac{E}{1+\nu}\varepsilon_{ij} + \frac{\nu E}{(1+\nu)(1-2\nu)}\varepsilon_{kk}\delta_{ij} \tag{2-73}$$

Upon expansion, Eq. (2-73) gives three equations for normal stress and six equations for shear stress. Equation (2-72) is often written in a briefer form by letting

$$\frac{\nu E}{(1+\nu)(1-2\nu)} = \lambda \quad \text{Lamé's constant}$$

and noting that $\Delta = \varepsilon_x + \varepsilon_y + \varepsilon_z$.

$$\sigma_x = 2G\varepsilon_x + \lambda\Delta \tag{2-74}$$

The stresses and the strains can be broken into deviator and hydrostatic components. The deviatoric response (distortion) is related to the stress deviator by

$$\sigma'_{ij} = \frac{E}{1+\nu}\varepsilon'_{ij} = 2G\varepsilon'_{ij} \tag{2-75}$$

while the relationship between hydrostatic stress and mean strain is

$$\sigma_{ii} = \frac{E}{1-2\nu}\varepsilon_{kk} = 3K\varepsilon_{kk} \tag{2-76}$$

For a case of *plane stress* ($\sigma_3 = 0$), two simple and useful equations relating stress to strain may be obtained by solving simultaneously two of the equations of (2-64).

$$\sigma_1 = \frac{E}{1-\nu^2}(\varepsilon_1 + \nu\varepsilon_2)$$
$$\sigma_2 = \frac{E}{1-\nu^2}(\varepsilon_2 + \nu\varepsilon_1) \tag{2-77}$$

A situation of plane stress exists typically in a thin sheet loaded in the plane of the sheet or a thin-wall tube loaded by internal pressure where there is no stress normal to a free surface.

Another important situation is *plane strain* ($\varepsilon_3 = 0$), which occurs typically when one dimension is much greater than the other two, as in a long rod or a cylinder with restrained ends. Some type of physical restraint exists to limit the strain in one direction, so

$$\varepsilon_3 = \frac{1}{E}\left[\sigma_3 - \nu(\sigma_1 + \sigma_2)\right] = 0$$

but

$$\sigma_3 = \nu(\sigma_1 + \sigma_2)$$

Therefore, a stress exists even though the strain is zero. Substituting this value into Eq. (2-64), we get

$$\varepsilon_1 = \frac{1}{E}\left[(1-\nu^2)\sigma_1 - \nu(1+\nu)\sigma_2\right]$$
$$\varepsilon_2 = \frac{1}{E}\left[(1-\nu^2)\sigma_2 - \nu(1+\nu)\sigma_1\right] \tag{2-78}$$
$$\varepsilon_3 = 0$$

Example Strain-gage measurements made on the free surface of a steel plate indicate that the principal strains are 0.004 and 0.001. What are the principal stresses?

Since this is a condition of plane stress, Eqs. (2-77) apply. From Table 2-1, $E = 200$ GPa and $\nu = 0.33$.

$$\sigma_1 = \frac{E}{1-\nu^2}(\varepsilon_1 + \nu\varepsilon_2) = \frac{200}{1-0.109}\{0.004 + 0.33(0.001)\}$$
$$= \frac{200}{0.891}(0.004 + 0.0003) = 0.965 \text{ GPa} = 965 \text{ MPa}$$
$$\sigma_2 = \frac{E}{1-\nu^2}(\varepsilon_2 + \nu\varepsilon_1) = \frac{200}{0.891}(0.001 + 0.0013) = 0.516 \text{ GPa}$$

Note the error that would result if the principal stresses were computed by simply multiplying Young's modulus by the strain.

$$\sigma_1 = E\varepsilon_1 = 200(0.004) = 800 \text{ MPa} \quad \text{incorrect}$$
$$\sigma_2 = E\varepsilon_1 = 200(0.001) = 200 \text{ MPa} \quad \text{incorrect}$$

2-13 STRAIN ENERGY

The *elastic strain energy* U is the energy expended by the action of external forces in deforming an elastic body. Essentially all the work performed during elastic deformation is stored as elastic energy, and this energy is recovered on the release of the applied forces. Energy (or work) is equal to a force multiplied by the distance over which it acts. In the deformation of an elastic body, the force and deformation increase linearly from initial values of zero so that the average energy is equal to one-half of their product. This is also equal to the area under the load-deformation curve.

$$U = \tfrac{1}{2}P\delta$$

For an elemental cube that is subjected to only a tensile stress along the x axis, the elastic strain energy is given by

$$dU = \tfrac{1}{2}P\,du = \tfrac{1}{2}(\sigma_x A)(\varepsilon_x\,dx)$$
$$= \tfrac{1}{2}(\sigma_x\varepsilon_x)(A\,dx) \tag{2-79}$$

Equation (2-79) describes the total elastic energy absorbed by the element. Since $A\,dx$ is the volume of the element, the *strain energy per unit volume* or strain energy density U_0 is given by

$$U_0 = \frac{1}{2}\sigma_x\varepsilon_x = \frac{1}{2}\frac{\sigma_x^2}{E} = \frac{1}{2}\varepsilon_x^2 E \tag{2-80}$$

Note that the lateral strains which accompany deformation in simple tension do not enter into the expression for strain energy because forces do not exist in the direction of the lateral strains.

By the same type of reasoning, the strain energy per unit volume of an element subjected to *pure shear* is given by

$$U_0 = \frac{1}{2}\tau_{xy}\gamma_{xy} = \frac{1}{2}\frac{\tau_{xy}^2}{G} = \frac{1}{2}\gamma_{xy}^2 G \tag{2-81}$$

The elastic strain energy for a general three-dimensional stress distribution may be obtained by superposition.

$$U_0 = \tfrac{1}{2}(\sigma_x\varepsilon_x + \sigma_y\varepsilon_y + \sigma_z\varepsilon_z + \tau_{xy}\gamma_{xy} + \tau_{xz}\gamma_{xz} + \tau_{yz}\gamma_{yz}) \tag{2-82}$$

or in tensor notation

$$U_0 = \tfrac{1}{2}\sigma_{ij}\varepsilon_{ij} \tag{2-83}$$

Substituting the equations of Hooke's law [Eqs. (2-64) and (2-65)] for the strains

in Eq. (2-82) results in an expression for strain energy per unit volume expressed solely in terms of the stress and the elastic constants

$$U_0 = \frac{1}{2E}\left(\sigma_x^2 + \sigma_y^2 + \sigma_z^2\right) - \frac{\nu}{E}\left(\sigma_x\sigma_y + \sigma_y\sigma_z + \sigma_x\sigma_z\right)$$
$$+ \frac{1}{2G}\left(\tau_{xy}^2 + \tau_{xz}^2 + \tau_{yz}^2\right) \tag{2-84}$$

Also, by substituting Eqs. (2-74) into Eq. (2-82), the stresses are eliminated, and the strain energy is expressed in terms of strains and the elastic constants

$$U_0 = \tfrac{1}{2}\lambda\Delta^2 + G\left(\varepsilon_x^2 + \varepsilon_y^2 + \varepsilon_z^2\right) + \tfrac{1}{2}G\left(\gamma_{xy}^2 + \gamma_{xz}^2 + \gamma_{yz}^2\right) \tag{2-85}$$

It is interesting to note that the derivative of U_0 with respect to any strain component gives the corresponding stress component. For example,

$$\frac{\partial U_0}{\partial\varepsilon_x} = \lambda\Delta + 2G\varepsilon_x = \sigma_x \tag{2-86}$$

In the same way, $\partial U_0/\partial\sigma_x = \varepsilon_x$. Methods of calculation using strain energy to arrive at stresses and strains are powerful tools in elasticity analysis. Some of the better known techniques are Castigliano's theorem, the theorem of least work, and the principal of virtual work.

2-14 ANISOTROPY OF ELASTIC BEHAVIOR

Up to this point we have considered elastic behavior from a simple phenomenological point of view, i.e., Hooke's law was presented as a well-established empirical law and our attention was directed at developing useful relationships between stress and strain in an isotropic elastic solid. In this section we consider the fact that the elastic constants of a crystal vary markedly with orientation. However, first it is important to discuss briefly the nature of the elastic forces between atoms.

When a force is applied to a crystalline solid, it either pulls the atoms apart or pushes them together. The applied force is resisted by the forces of attraction or repulsion between the atoms. A convenient way to look at this is with an energy-distance diagram (Fig. 2-19), which represents the interaction energy (potential energy) between two atoms as they are separated by a distance a. When the external force is zero, the atoms are separated by a distance equal to the equilibrium spacing $a = a_0$. For small applied forces, the atoms will find a new equilibrium spacing a at which the external and internal forces are balanced. The displacement of the atom is $u = a - a_0$. Since force is the derivative of potential energy with distance [compare Eq. (2-86)], the force to produce a given equilibrium displacement is

$$P = \frac{d\phi(u)}{du} \tag{2-87}$$

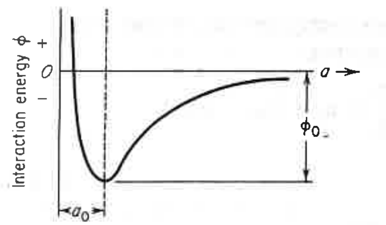


Figure 2-19 Interaction energy vs. separation between atoms

where $\phi(u)$ is the interaction bond energy at a displacement u. Thus, the force on a bond is a function of displacement u. For each displacement there is a characteristic value of force $P(u)$. Moreover, the deformation of the bonds between atoms is reversible. When the displacement returns to some initial value u_1 after being extended to u_2 the force returns to its previous value $P(u_1)$.

In an elastic solid the bond energy is a continuous function of displacement.[1] Thus, we can express $\phi(u)$ as a Taylor series

$$\phi(u) = \phi_0 + \left(\frac{d\phi}{du}\right)_0 u + \frac{1}{2}\left(\frac{d^2\phi}{du^2}\right)_0 u^2 + \cdots \qquad (2\text{-}88)$$

where ϕ_0 is the energy at $u = 0$ and the differential coefficients are measured at $u = 0$. Since the force is zero when $a = a_0$, $d\phi/du = 0$

$$\phi(u) = \phi_0 + \frac{1}{2}\left(\frac{d^2\phi}{du^2}\right)_0 u^2$$
$$P = \frac{d\phi(u)}{du} = \left(\frac{d^2\phi}{du^2}\right)_0 u \qquad (2\text{-}89)$$

The coefficient $(d^2\phi/du^2)_0$ is the curvature of the energy-distance curve at $u = a_0$. Since it is independent of u, the coefficient is a constant, and Eq. (2-89) is equivalent to $P = ku$, which is Hooke's law in its original form. When Eq. (2-89) is expressed in terms of stress and strain, the coefficient is directly proportional to the elastic constant of the material. It has the same value for both tension and compression since it is independent of the sign of u. Thus, we have shown that the elastic constant is determined by the sharpness of curvature of the minimum in the energy-distance curve. It is therefore a basic property of the material, not readily changed by heat treatment or defect structure, although it would be expected to decrease with increasing temperature. Moreover, since the binding forces will be strongly affected by distance between atoms, the elastic constants will vary with direction in the crystal lattice.

[1] This development follows that given by A. H. Cottrell, "The Mechanical Properties of Matter," pp. 84–85, John Wiley & Sons, Inc., New York, 1964.

In the generalized case[1] Hooke's law may be expressed as

$$\varepsilon_{ij} = S_{ijkl}\sigma_{kl} \qquad (2\text{-}90)$$

and

$$\sigma_{ij} = C_{ijkl}\varepsilon_{kl} \qquad (2\text{-}91)$$

where S_{ijkl} is the *compliance tensor* and C_{ijkl} is the *elastic stiffness* (often called just the elastic constants). Both S_{ijkl} and C_{ijkl} are fourth-rank tensor quantities. If we expanded Eq. (2-90) or (2-91), we would get nine equations, each with nine terms, 81 constants in all. However, we know that both ε_{ij} and σ_{ij} are symmetric tensors, that is, $\sigma_{ij} = \sigma_{ji}$, which immediately leads to appreciable simplification. Thus, we can write

$$\varepsilon_{ij} = S_{ijkl}\sigma_{kl} \quad \text{or} \quad \varepsilon_{ij} = S_{ijlk}\sigma_{lk}$$

and since

$$S_{ijkl}\sigma_{kl} = S_{ijlk}\sigma_{lk}$$
$$\sigma_{kl} = \sigma_{lk} \quad \text{and} \quad S_{ijkl} = S_{ijlk}$$

Also, we could write

$$\varepsilon_{ij} = S_{ijkl}\sigma_{kl} = \varepsilon_{ji} = S_{jikl}\sigma_{kl}$$
$$S_{ijkl} = S_{jikl}$$

Therefore, because of the symmetry of the stress and strain tensors, only 36 of the components of the compliance tensor are independent and distinct terms. The same is true of the elastic stiffness tensor.

Expanding Eq. (2-91) and taking into account the above relationships gives equations like

$$\sigma_{11} = C_{1111}\varepsilon_{11} + C_{1122}\varepsilon_{22} + C_{1133}\varepsilon_{33} + C_{1123}(2\varepsilon_{23}) + C_{1113}(2\varepsilon_{13}) + C_{1112}(2\varepsilon_{12})$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$
$$\sigma_{23} = C_{2311}\varepsilon_{11} + C_{2322}\varepsilon_{22} + C_{2333}\varepsilon_{33} + C_{2323}(2\varepsilon_{23}) + C_{2313}(2\varepsilon_{13}) + C_{2312}(2\varepsilon_{12})$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \qquad (2\text{-}92)$$

These equations show that, in contrast to the situation for an isotropic elastic solid, Eq. (2-72), for an anisotropic elastic solid both normal strains and shear strains are capable of contributing to a normal stress.

[1] An excellent text that deals with the anisotropic properties of crystals in tensor notation is J. F. Nye, "Physical Properties of Crystals," Oxford University Press, London, 1957. For a treatment of anisotropic elasticity see R. F. S. Hearmon, "An Introduction to Applied Anisotropic Elasticity," Oxford University Press, London, 1961. A fairly concise but complete discussion of crystal elasticity is given by S. M. Edelglass, "Engineering Materials Science," pp. 277–301. The Ronald Press Company, New York, 1966.

In expanding Eq. (2-90), we express the shearing strains by the more conventional engineering shear strain $\gamma = 2\varepsilon$.

$$\begin{aligned}
\varepsilon_{11} &= S_{1111}\sigma_{11} + S_{1122}\sigma_{22} + S_{1133}\sigma_{33} + 2S_{1123}\sigma_{23} + 2S_{1113}\sigma_{13} + 2S_{1112}\sigma_{12} \\
&\cdots\cdots\cdots \\
&\cdots\cdots\cdots \\
\gamma_{23} = 2\varepsilon_{23} &= 2S_{2311}\sigma_{11} + 2S_{2322}\sigma_{22} + 2S_{2333}\sigma_{33} + 4S_{2323}\sigma_{23} \\
&\qquad + 4S_{2313}\sigma_{13} + 4S_{2312}\sigma_{12} \\
&\cdots\cdots\cdots \\
&\cdots\cdots\cdots
\end{aligned} \tag{2-93}$$

The usual convention for designating components of elastic compliance and elastic stiffness uses only two subscripts instead of four. This is called the *contracted notation*. The subscripts simply denote the row and column in the matrix of components in which they fall.

$$\begin{aligned}
\sigma_{11} &= C_{11}\varepsilon_{11} + C_{12}\varepsilon_{22} + C_{13}\varepsilon_{33} + C_{14}\gamma_{23} + C_{15}\gamma_{13} + C_{16}\gamma_{12} \\
&\cdots\cdots\cdots \\
&\cdots\cdots\cdots \\
\sigma_{23} &= C_{41}\varepsilon_{11} + C_{42}\varepsilon_{22} + C_{43}\varepsilon_{33} + C_{44}\gamma_{23} + C_{45}\gamma_{13} + C_{46}\gamma_{12} \\
&\cdots\cdots\cdots \\
&\cdots\cdots\cdots
\end{aligned} \tag{2-94}$$

and

$$\begin{aligned}
\varepsilon_{11} &= S_{11}\sigma_{11} + S_{12}\sigma_{22} + S_{13}\sigma_{33} + S_{14}\sigma_{23} + S_{15}\sigma_{13} + S_{16}\sigma_{12} \\
&\cdots\cdots\cdots \\
&\cdots\cdots\cdots \\
\sigma_{23} &= S_{41}\sigma_{11} + S_{42}\sigma_{22} + S_{43}\sigma_{33} + S_{44}\sigma_{23} + S_{45}\sigma_{13} + S_{46}\sigma_{12} \\
&\cdots\cdots\cdots \\
&\cdots\cdots\cdots
\end{aligned} \tag{2-95}$$

By comparing coefficients in Eqs. (2-92) and (2-94) and Eqs. (2-93) and (2-95) we note, for example, that

$$C_{2322} = C_{42} \qquad C_{1122} = C_{12}$$

$$S_{1122} = C_{12} \qquad 2S_{2311} = C_{41} \qquad 4S_{2323} = S_{44}$$

The elastic stiffness constants are defined by equations like

$$C_{11} = \frac{\Delta\sigma_{11}}{\Delta\varepsilon_{11}} \qquad \text{all } \varepsilon_{ij} \text{ constant except } \varepsilon_{11}$$

Unfortunately, a measurement such as this is difficult to do experimentally since

the specimen must be constrained mechanically to prevent strains such as ε_{23}. It is much easier to experimentally determine the coefficients of the elastic compliance from equations of the type

$$S_{11} = \frac{\Delta\varepsilon_{11}}{\Delta\sigma_{11}} \qquad \text{all } \sigma_{ij} \text{ constant except } \sigma_{11}$$

If the components of S_{ij} have been determined experimentally, then the components of C_{ij} can be determined by matrix inversion.

At this stage we have 36 independent constants, but further reduction in the number of independent constants is possible. By using the relationship given in Eq. (2-86), we can show that the constants are symmetrical, that is, $C_{ij} = C_{ji}$. For example,

$$\frac{\partial U}{\partial \varepsilon_{11}} = \sigma_{11} = C_{11}\varepsilon_{11} + C_{12}\varepsilon_{22} + C_{13}\varepsilon_{33} + C_{14}\gamma_{23} + C_{15}\gamma_{13} + C_{16}\gamma_{12}$$

$$\frac{\partial^2 U}{\partial \varepsilon_{11}\,\partial \varepsilon_{22}} = C_{12}$$

$$\frac{\partial U}{\partial \varepsilon_{22}} = \sigma_{22} = C_{21}\varepsilon_{11} + C_{22}\varepsilon_{22} + C_{23}\varepsilon_{33} + C_{24}\gamma_{23} + C_{25}\gamma_{13} + C_{26}\gamma_{12}$$

$$\frac{\partial^2 U}{\partial \varepsilon_{22}\,\partial \varepsilon_{11}} = C_{21}$$

$$\therefore \quad \frac{\partial^2 U}{\partial \varepsilon_{11}\,\partial \varepsilon_{22}} = \frac{\partial^2 U}{\partial \varepsilon_{22}\,\partial \varepsilon_{11}} = C_{12} = C_{21}$$

In general, $C_{ij} = C_{ji}$ and $S_{ij} = S_{ji}$. Now, we start with 36 constants C_{ij}, but of these there are six consants where $i = j$. This leaves 30 constants where $i \neq j$, but only one-half of these are independent constants since $C_{ij} = C_{ji}$. Therefore, for the general anisotropic linear elastic solid there are $30/2 + 6 = 21$ independent elastic constants.

As a result of symmetry conditions found in different crystal structures the number of independent elastic constants can be reduced still further.

Crystal structure	Rotational symmetry	Number of independent elastic constants
Triclinic	None	21
Monoclinic	1 twofold rotation	13
Orthorhombic	2 perpendicular twofold rotations	9
Tetragonal	1 fourfold rotation	6
Hexagonal	1 sixfold rotation	5
Cubic	4 threefold rotations	3
Isotropic		2

Table 2-2 Stiffness and compliance constants for cubic crystals

Metal	C_{11}	C_{12}	C_{44}	S_{11}	S_{12}	S_{44}
Aluminum	108.2	61.3	28.5	15.7	−5.7	35.1
Copper	168.4	121.4	75.4	14.9	−6.2	13.3
Iron	237.0	141.0	116.0	8.0	−2.8	8.6
Tungsten	501.0	198.0	151.4	2.6	−0.7	6.6

Stiffness constants in units of GPa.
Compliance constants in units of TPa^{-1}.

For a cubic crystal structure

$$C_{11} = \frac{S_{11} + S_{12}}{(S_{11} - S_{12})(S_{11} + 2S_{12})}$$

$$C_{12} = \frac{-S_{12}}{(S_{11} - S_{12})(S_{11} + 2S_{12})} \tag{2-96}$$

$$C_{44} = \frac{1}{S_{44}}$$

The modulus of elasticity in any direction of a cubic crystal (described by the direction cosines l, m, n) is given by

$$\frac{1}{E} = S_{11} - 2\left[(S_{11} - S_{12}) - \frac{1}{2}S_{44}\right](l^2m^2 + m^2n^2 + l^2n^2) \tag{2-97}$$

Typical values of elastic constants for cubic metals are given in Table 2-2.

By comparing the generalized Hooke's law Eqs. (2-95) with the equations using the common technical moduli Eq. (2-64) we can conclude that the elastic constants for an isotropic material are given by

$$S_{11} = \frac{1}{E} \qquad S_{12} = -\frac{\nu}{E} \qquad S_{44} = \frac{1}{G}$$

Since S_{11} and S_{12} are the independent constants, their relationship to S_{44} can be obtained from Eq. (2-68)

$$G = \frac{E}{2(1 + \nu)} = \frac{1}{2(1/E + \nu/E)}$$

$$G = \frac{1}{S_{44}} = \frac{1}{2(S_{11} - S_{12})}$$

or

$$S_{44} = 2(S_{11} - S_{12}) \tag{2-98}$$

Comparable equations relating the elastic stiffness constants can be developed

from Eqs. (2-95) and (2-74).

$$C_{12} = \lambda \quad \text{Lamé's constant}$$

$$C_{11} = 2G + \lambda \tag{2-99}$$

$$C_{44} = \tfrac{1}{2}(C_{11} - C_{12})$$

The technical elastic moduli E, ν, and G are usually measured by direct static measurements in the tension or torsion tests. However, where more precise measurements are required or where measurements are required in small single-crystal specimens cut along specified directions, dynamic techniques using measurement of frequency or elapsed time are frequently employed. Dynamic measurements involve very small atomic displacements and low stresses compared with static modulus measurements. The velocity of propagation of a displacement down a cylindrical-crystal specimen is given by

$$v_x = \frac{\omega\lambda}{2\pi}\sqrt{\frac{E_x}{\rho}} \tag{2-100}$$

where ω is the natural frequency of vibration of a stress pulse of wavelength λ in a crystal of density ρ. Dynamic techniques consist of measuring either the natural frequency of vibration or the elapsed time for an ultrasonic pulse to travel down the specimen and return. Because the strain cycles produced in dynamic testing occur at high rates, there is very little time for heat transfer to take place. Thus, dynamic measurements of elastic constants are obtained under adiabatic conditions, while static elastic measurements are obtained under essentially isothermal conditions. There is a small difference between adiabatic and isothermal elastic moduli.[1]

$$E_{\text{adi}} = \frac{E_{\text{iso}}}{1 - \dfrac{E_{\text{iso}}T\alpha^2}{9c}} \tag{2-101}$$

where α is the volume coefficient of thermal expansion and c is the specific heat. Since the specific heat of a solid is large compared to a gas, the difference between adiabatic and isothermal moduli is not great and can be ignored for practical purposes.

Example Determine the modulus of elasticity for tungsten and iron in the ⟨111⟩ and ⟨100⟩ directions. What conclusions can be drawn about their elastic anisotropy? From Table 2-2

	S_{11}	S_{12}	S_{44}
Fe:	8.0	−2.8	8.6
W:	2.6	−0.7	6.6

[1] For a derivation of Eq. (2-101) see S. M. Edelgass, op. cit., pp. 294–297.

The direction cosines for the chief directions in a cubic lattice are:

Directions	l	m	n
⟨100⟩	1	0	0
⟨110⟩	$1/\sqrt{2}$	$1/\sqrt{2}$	0
⟨111⟩	$1/\sqrt{3}$	$1/\sqrt{3}$	$1/\sqrt{3}$

For iron:

$$\frac{1}{E_{111}} = 8.0 - 2\{(8.0 + 2.8) - 8.6/2\}\left(\frac{1}{9} + \frac{1}{9} + \frac{1}{9}\right)$$

$$\frac{1}{E_{111}} = 8.0 - 2(10.8 - 4.3)\left(\frac{1}{3}\right) = 8.0 - 13.0\left(\frac{1}{3}\right)$$

$$= 8.0 - 4.3 = 3.7 \text{ TPa}^{-1}$$

$$E_{111} = \frac{1}{3.7} \text{ TPa} = 270 \text{ GPa}$$

$$\frac{1}{E_{100}} = 8.0 - 13.0(0) = 8.0 \text{ TPa}^{-1} \qquad E_{100} = 125 \text{ GPa}$$

For tungsten:

$$\frac{1}{E_{111}} = 2.6 - 2\left\{(2.6 + 0.7) - \frac{6.6}{2}\right\}\left(\frac{1}{3}\right)$$

$$\frac{1}{E_{111}} = 2.6 - 2\{3.3 - 3.3\}\left(\frac{1}{3}\right) = 2.6 \text{ TPa}^{-1}$$

$$E_{111} = \frac{1}{0.26} \text{ TPa} = 385 \text{ GPa}$$

$$\frac{1}{E_{100}} = 2.6 - 2\left\{(2.6 + 0.7) - \frac{6.6}{2}\right\}(0) = 2.6 \text{ TPa}^{-1}$$

$$E_{100} = \frac{1}{0.26} \text{ TPa} = 385 \text{ GPa}$$

Therefore, we see that tungsten is elastically isotropic while iron is elastically anisotropic.

2-15 STRESS CONCENTRATION

A geometrical discontinuity in a body, such as a hole or a notch, results in a nonuniform stress distribution at the vicinity of the discontinuity. At some region near the discontinuity the stress will be higher than the average stress at distances removed from the discontinuity. Thus, a *stress concentration* occurs at the

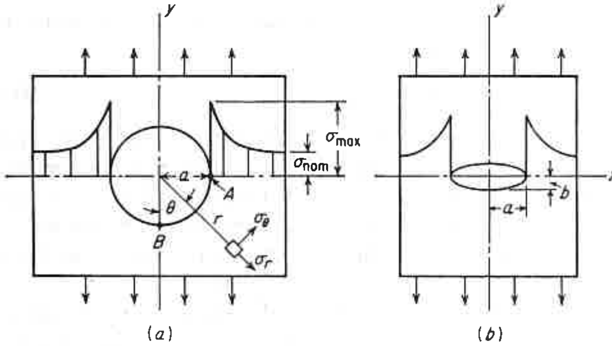


Figure 2-20 Stress distributions due to (*a*) circular hole and (*b*) elliptical hole.

discontinuity, or *stress raiser*. Figure 2-20*a* shows a plate containing a circular hole which is subjected to a uniaxial load. If the hole were not present, the stress would be uniformly distributed over the cross section of the plate and it would be equal to the load divided by the cross-sectional area of the plate. With the hole present, the distribution is such that the axial stress reaches a high value at the edges of the hole and drops off rapidly with distance away from the hole.

The stress concentration is expressed by a theoretical stress-concentration factor K_t. Generally K_t is described as the ratio of the maximum stress to the nominal stress based on the net section, although some workers use a value of nominal stress based on the entire cross section of the member in a region where there is no stress concentrator.

$$K_t = \frac{\sigma_{\max}}{\sigma_{\text{nominal}}} \tag{2-102}$$

In addition to producing a stress concentration, a notch also creates a localized condition of biaxial or triaxial stress. For example, for the circular hole in a plate subjected to an axial load, a radial stress is produced as well as a longitudinal stress. From elastic analysis,[1] the stresses produced in an infinitely wide plate containing a circular hole and axially loaded can be expressed as

$$\sigma_r = \frac{\sigma}{2}\left(1 - \frac{a^2}{r^2}\right) + \frac{\sigma}{2}\left(1 + 3\frac{a^4}{r^4} - 4\frac{a^2}{r^2}\right)\cos 2\theta$$

$$\sigma_\theta = \frac{\sigma}{2}\left(1 + \frac{a^2}{r^2}\right) - \frac{\sigma}{2}\left(1 + 3\frac{a^4}{r^4}\right)\cos 2\theta \tag{2-103}$$

$$\tau = -\frac{\sigma}{2}\left(1 - 3\frac{a^4}{r^4} + 2\frac{a^2}{r^2}\right)\sin 2\theta$$

[1] Timoshenko and Goodier, *op. cit.*, pp. 78–81.

3.4 YIELDING CRITERIA FOR DUCTILE METALS

The problem of deducing mathematical relationships for predicting the conditions at which plastic yielding begins when a material is subjected to any possible combination of stresses is an important consideration in the field of plasticity. In uniaxial loading, as in a tension test, macroscopic plastic flow begins at the yield stress σ_0. It is expected that yielding under a situation of combined stresses can be related to some particular combination of principal stresses. There is at present no theoretical way of calculating the relationship between the stress components to correlate yielding for a three-dimensional state of stress with yielding in the uniaxial tension test.

The yielding criteria are essentially empirical relationships. However, a yield criterion must be consistent with a number of experimental observations, the chief of which is that pure hydrostatic pressure does not cause yielding in a continuous solid.[1] As a result of this, the hydrostatic component of a complex state of stress does not influence the stress at which yielding occurs. Therefore, we look for the stress deviator to be involved with yielding. Moreover, for an isotropic material, the yield criterion must be independent of the choice of axes, i.e., it must be an invariant function. These considerations lead to the conclusion that the yield criteria must be some function of the invariants of the stress deviator. At present there are two generally accepted criteria for predicting the onset of yielding in ductile metals.

Von Mises' or Distortion-Energy Criterion

Von Mises (1913) proposed that yielding would occur when the second invariant of the stress deviator J_2 exceeded some critical value.

$$J_2 = k^2 \tag{3-10}$$

where $J_2 = \frac{1}{6}[(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2]$.

To evaluate the constant k and relate it to yielding in the tension test, we realize that at yielding in uniaxial tension $\sigma_1 = \sigma_0$, $\sigma_2 = \sigma_3 = 0$

$$\sigma_0^2 + \sigma_0^2 = 6k^2$$

$$\sigma_0 = \sqrt{3}\,k \tag{3-11}$$

Substituting Eq. (3-11) in Eq. (3-10) results in the usual form of the von Mises' yield criterion

$$\sigma_0 = \frac{1}{\sqrt{2}}\left[(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2\right]^{1/2} \tag{3-12}$$

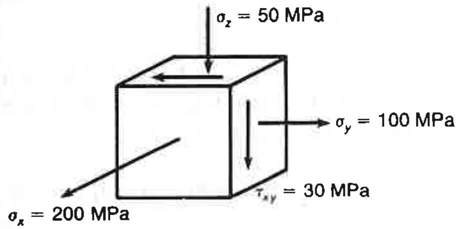
[1] A significant influence of hydrostatic or mean stress of modest values on yielding has been observed in glassy polymers such as PMMA. S. S. Sternstein and L. Ongchin, *Polym. Prepr. Am. Chem. Soc. Div. Polym. Chem.*, September 1969.

or from Eq. (2-61)

$$\sigma_0 = \frac{1}{\sqrt{2}}\left[(\sigma_x - \sigma_y)^2 + (\sigma_y - \sigma_z)^2 + (\sigma_z - \sigma_x)^2 + 6\left(\tau_{xy}^2 + \tau_{yz}^2 + \tau_{xz}^2\right)\right]^{1/2} \tag{3-13}$$

Equation (3-12) or (3-13) predicts that yielding will occur when the differences of stresses on the right side of the equation exceed the yield stress in uniaxial tension σ_0.

Example Stress analysis of a spacecraft structural member gives the state of stress shown below. If the part is made from 7075-T6 aluminum alloy with $\sigma_0 = 500$ MPa, will it exhibit yielding? If not, what is the safety factor?



From Eq. (3-13)

$$\sigma_0 = \frac{1}{\sqrt{2}}\left[(200 - 100)^2 + (100 - (-50))^2 + (-50 - 200)^2 + 6(30)^2\right]^{1/2}$$

$$\sigma_0 = \frac{1}{\sqrt{2}}(100{,}400)^{1/2} = \frac{316.859}{\sqrt{2}} = 224 \text{ MPa}$$

Since the value of σ_0 calculated from the yield criterion is less than the yield strength of the aluminum alloy, yielding will not occur. The safety factor is $500/224 = 2.2$.

To identify the constant k in Eq. (3-10), consider the state of stress in pure shear, as is produced in a torsion test.

$$\sigma_1 = -\sigma_3 = \tau \qquad \sigma_2 = 0$$

$$\text{at yielding} \qquad \sigma_1^2 + \sigma_1^2 + 4\sigma_1^2 = 6k^2$$

$$\therefore \quad \sigma_1 = k$$

so that k represents the yield stress in pure shear (torsion). Therefore, the von Mises' criterion predicts that the yield stress in torsion will be less than in uniaxial tension according to

$$k = \frac{1}{\sqrt{3}}\sigma_0 = 0.577\sigma_0 \tag{3-14}$$

To summarize, note that the von Mises' yield criterion implies that yielding is not dependent on any particular normal stress or shear stress, but instead, yielding depends on a function of all three values of principal shearing stress. Since the yield criterion is based on differences of normal stresses, $\sigma_1 - \sigma_2$, etc., the criterion is independent of the component of hydrostatic stress. Since the von Mises' yield criterion involves squared terms, the result is independent of the sign of the individual stresses. This is an important advantage since it is not necessary to know which are the largest and smallest principal stresses in order to use this yield criterion.

Von Mises originally proposed this criterion because of its mathematical simplicity. Subsequently, other workers have attempted to give it physical meaning. Hencky (1924) showed that Eq. (3-12) was equivalent to assuming that yielding occurs when the *distortion energy* reaches a critical value. The distortion energy is that part of the total strain energy per unit volume that is involved in change of shape as opposed to a change in volume.

Example The fact that the total strain energy can be split into a term depending on change of volume and a term depending on distortion can be seen by expressing Eq. (2-84) in terms of principal stresses.

$$U_0 = \frac{1}{2E}\left[\sigma_1^2 + \sigma_2^2 + \sigma_3^2 - 2\nu(\sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_1\sigma_3)\right] \tag{3-15}$$

or expressing in terms of the invariants of the stress tensor

$$U_0 = \frac{1}{2E}\left[I_1^2 - 2I_2(1 + \nu)\right] \tag{3-16}$$

This equation is more meaningful if we express it in terms of the bulk modulus (volume change) and the shear modulus (distortion). From Sec. 2-11,

$$E = \frac{9GK}{3K + G} \qquad \nu = \frac{3K - 2G}{6K + 2G}$$

Substituting into Eq. (3-16)

$$U_0 = \frac{I_1^2}{18K} + \frac{1}{6G}\left(I_1^2 - 3I_2\right) \tag{3-17}$$

Equation (3-17) is important because it shows that the total strain energy can be split into a term depending on change of volume and a term depending on distortion.

$$(U_0)_{\text{distortion}} = \frac{1}{6G}\left(\sigma_1^2 + \sigma_2^2 + \sigma_3^2 - \sigma_1\sigma_2 - \sigma_2\sigma_3 - \sigma_1\sigma_3\right)$$

or

$$(U_0)_{\text{distortion}} = \frac{1}{12G}\left[(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2\right] \tag{3-18}$$

For a uniaxial state of stress, $\sigma_1 = \sigma_0$, $\sigma_2 = \sigma_3 = 0$

$$(U_0)_{\text{distortion}} = \frac{1}{12G}2\sigma_0^2$$

or

$$\sigma_0 = \frac{1}{\sqrt{2}}\left[(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2\right]^{1/2} \tag{3-19}$$

Another physical interpretation given to the von Mises' yield criterion is that it represents the critical value of the octahedral shear stress (see Sec. 3-9). This is the shear stress on the octahedral planes which make equal angles with the principal axes. Still another interpretation is that it represents the mean square of the shear stress averaged over all orientations in the solid.[1]

Maximum-Shear-Stress or Tresca Criterion

This yield criterion assumes that yielding occurs when the maximum shear stress reaches the value of the shear stress in the uniaxial-tension test. From Eq. (2-21), the maximum shear stress is given by

$$\tau_{\max} = \frac{\sigma_1 - \sigma_3}{2} \tag{3-20}$$

where σ_1 is the algebraically largest and σ_3 is the algebraically smallest principal stress.

For uniaxial tension, $\sigma_1 = \sigma_0$, $\sigma_2 = \sigma_3 = 0$, and the shearing yield stress τ_0 is equal to $\sigma_0/2$. Substituting in Eq. (3-20),

$$\tau_{\max} = \frac{\sigma_1 - \sigma_3}{2} = \tau_0 = \frac{\sigma_0}{2}$$

Therefore, the maximum-shear-stress criterion is given by

$$\sigma_1 - \sigma_3 = \sigma_0 \tag{3-21}$$

For a state of pure shear, $\sigma_1 = -\sigma_3 = k$, $\sigma_2 = 0$, the maximum-shear-stress criterion predicts that yielding will occur when

$$\sigma_1 - \sigma_3 = 2k = \sigma_0$$

or

$$k = \frac{\sigma_0}{2}$$

so that the maximum-shear-stress criterion may be written

$$\sigma_1 - \sigma_3 = \sigma_1' - \sigma_3' = 2k \tag{3-22}$$

We note that the maximum-shear-stress criterion is less complicated mathematically than the von Mises' criterion, and for this reason it is often used in engineering design. However, the maximum-shear criterion does not take into consideration the intermediate principal stress. It suffers from the major difficulty that it is necessary to know in advance which are the maximum and minimum principal stresses. Moreover, the general form of the maximum-shear-stress criterion, Eq. (3-23), is far more complicated than the von Mises' criterion, Eq.

[1] See G. Sines, "Elasticity and Strength," pp. 54–56, Allyn and Bacon, Inc., Boston, 1969.

(3-10), and for this reason the von Mises' criterion is preferred in most theoretical work.

$$4J_2^3 - 27J_3^2 - 36k^2J_2^2 + 96k^4J_2 - 64k^6 = 0 \tag{3-23}$$

Example Use the maximum-shear-stress criterion to establish whether yielding will occur for the stress state shown in the previous example.

$$\tau_{max} = \frac{\sigma_x - \sigma_z}{2} = \frac{\sigma_0}{2}$$

$$200 - (-50) = \sigma_0$$

$$\sigma_0 = 250 \text{ MPa}$$

Again, the calculated value of σ_0 is less than the yield strength of the material.

3-5 COMBINED STRESS TESTS

The conditions for yielding under states of stress other than uniaxial and torsion loading can be studied conveniently with thin-wall tubes. Axial stress can be combined with torsion to produce various combinations of shear stress to normal stress intermediate between the values obtained separately in tension and torsion. Alternatively, a hydrostatic pressure may be introduced to produce a circumferential hoop stress in the tube.[1]

For the stresses shown in Fig. 3-3, from Eq. (2-9) the principal stresses are

$$\sigma_1 = \frac{\sigma_x}{2} + \left(\frac{\sigma_x^2}{4} + \tau_{xy}^2\right)^{1/2}$$
$$\sigma_2 = 0 \tag{3-24}$$
$$\sigma_3 = \frac{\sigma_x}{2} - \left(\frac{\sigma_x^2}{4} + \tau_{xy}^2\right)^{1/2}$$

Therefore, the maximum-shear-stress criterion of yielding is given by

$$\left(\frac{\sigma_x}{\sigma_0}\right)^2 + 4\left(\frac{\tau_{xy}}{\sigma_0}\right)^2 = 1 \tag{3-25}$$

and the distortion-energy theory of yielding is expressed by

$$\left(\frac{\sigma_x}{\sigma_0}\right)^2 + 3\left(\frac{\tau_{xy}}{\sigma_0}\right)^2 = 1 \tag{3-26}$$

[1] See for example S. S. Hecker, *Metall. Trans.*, vol. 2, pp. 2077–2086, 1971. A unique method for determining the yield locus of a flat sheet has been presented by D. Lee and W. A. Backofen, *Trans. Metall. Soc. AIME*, vol. 236, pp. 1077–1084, 1966. This method is well suited for studying the anisotropy of rolled sheet.

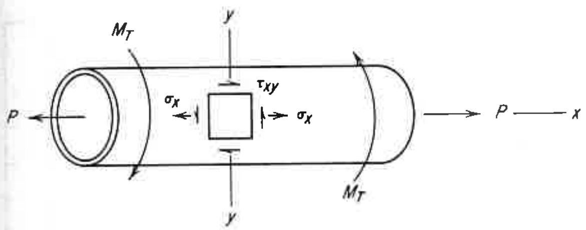


Figure 3-3 Combined tension and torsion in a thin-walled tube.

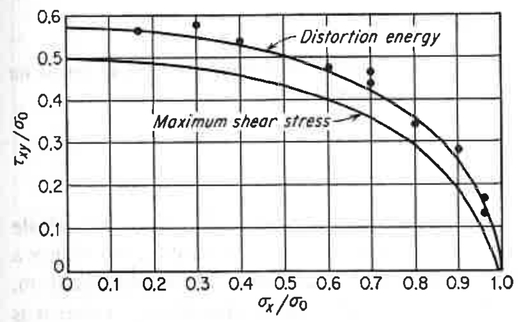


Figure 3-4 Comparison between maximum-shear-stress theory and distortion-energy (von Mises') theory.

Both equations define an ellipse. Figure 3-4 shows that the experimental results[1] agree best with the distortion-energy theory.

3-6 THE YIELD LOCUS

For a biaxial plane-stress condition ($\sigma_2 = 0$) the von Mises' yield criterion can be expressed mathematically as

$$\sigma_1^2 + \sigma_3^2 - \sigma_1\sigma_3 = \sigma_0^2 \tag{3-27}$$

This is the equation of an ellipse whose major semiaxis is $\sqrt{2}\,\sigma_0$ and whose minor semiaxis is $\sqrt{\frac{2}{3}}\,\sigma_0$. The plot of Eq. (3-27) is called a *yield locus* (Fig. 3-5). Several important points on the yield ellipse corresponding to particular stress-ratio loading paths are noted on the figure.

The yield locus for the maximum-shear-stress criterion falls inside of the von Mises' yield ellipse. Note that the two yielding criteria predict the same yield stress for conditions of uniaxial stress and balanced biaxial stress ($\sigma_1 = \sigma_3$). The greatest divergence between the two criteria occurs for pure shear ($\sigma_1 = -\sigma_3$). The yield stress predicted by the von Mises' criterion is 15.5 percent greater than the yield stress predicted by the maximum-shear-stress criterion.

[1] G. I. Taylor and H. Quinney, *Proc. R. Soc. London Ser. A.*, vol. 230A, pp. 323–362, 1931.

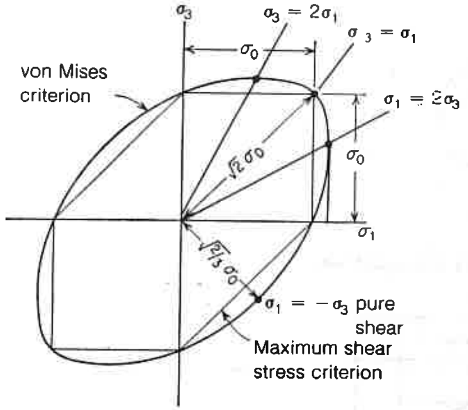


Figure 3-5 Comparison of yield criteria for plane stress.

3-7 ANISOTROPY IN YIELDING

The yielding criteria considered so far assume that the material is isotropic. While this may be the case at the start of plastic deformation, it certainly is no longer a valid assumption after the metal has undergone appreciable plastic deformation. Moreover, most fabricated metal shapes have anisotropic properties, so that it is likely that the tubular specimens used for basic studies of yield criteria incorporate some degree of anisotropy. Certainly the von Mises' criterion as formulated in Eq. (3-12) would not be valid for a highly oriented cold-rolled sheet or a fiber-reinforced composite material.

Hill[1] has formulated the von Mises' yield criterion for an anisotropic material having orthotropic symmetry.

$$F(\sigma_y - \sigma_z)^2 + G(\sigma_z - \sigma_x)^2 + H(\sigma_x - \sigma_y)^2 + 2L\tau_{yz}^2 + 2M\tau_{zx}^2 + 2N\tau_{xy}^2 = 1$$

where $F, G, \ldots, N$ are constants defining the degree of anisotropy. For principal axes of orthotropic symmetry

$$F(\sigma_2 - \sigma_3)^2 + G(\sigma_3 - \sigma_1)^2 + H(\sigma_1 - \sigma_2)^2 = 1 \qquad (3\text{-}28)$$

If X is the yield stress in the 1 direction, Y is the yield stress in the 2 direction, Z is the yield stress in the 3 direction, then by substituting into Eq. (3-28) we can evaluate the constants by

$$G + H = \frac{1}{X^2} \qquad H + F = \frac{1}{Y^2} \qquad F + G = \frac{1}{Z^2}$$

Lubahn and Felgar[2] give detailed plasticity calculations for anisotropic behavior.

[1] R. Hill, *Proc. R. Soc. London, Ser. B*, vol. 193, pp. 281–297, 1948.
[2] J. D. Lubahn and R. P. Felgar, "Plasticity and Creep of Metals," chap. 13, John Wiley & Sons, Inc., New York, 1961.

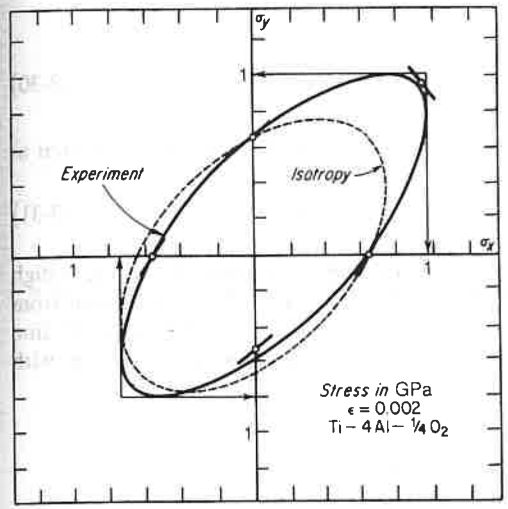


Figure 3-6 Yield locus for textured titanium-alloy sheet. *(After D. Lee and W. A. Backofen, Trans. Metall. Soc. AIME, vol. 236, p. 1083, 1966. By permission of the publishers.)*

On a plane-stress yield locus, such as Fig. 3-5, anisotropic yielding results in distortion of the yield ellipse. Figure 3-6 shows the yield locus for highly textured titanium alloy sheet.[1] Note that the experimentally determined curve is nonsymmetric when compared with the ideal isotropic curve.

An important aspect of yield anisotropy is *texture hardening*.[2] Consider a highly textured sheet that is fabricated into a thin-wall pressure vessel, so that the thickness stress σ_3 is negligible. From Eq. (3-28)

$$F\sigma_2^2 + G\sigma_1^2 + H(\sigma_1^2 - 2\sigma_1\sigma_2 + \sigma_2^2) = 1$$

$$(G + H)\sigma_1^2 + (F + H)\sigma_2^2 - 2H\sigma_1\sigma_2 = 1$$

or

$$\left(\frac{\sigma_1}{X}\right)^2 + \left(\frac{\sigma_2}{Y}\right)^2 - 2HXY\left(\frac{\sigma_1}{X}\frac{\sigma_2}{Y}\right) = 1 \qquad (3\text{-}29)$$

For simplicity, we shall assume that the yield stresses in the plane of the sheet are equal, that is, $X = Y$. Thus,

$$\sigma_1^2 + \sigma_2^2 - 2HY^2\sigma_1\sigma_2 = Y^2$$

and

$$G = F = \frac{1}{2Z^2} \qquad HY^2 = 1 - \frac{1}{2}\left(\frac{Y}{Z}\right)^2$$

However, the yield stress in the thickness direction of the sheet, Z, is a difficult property to measure. This problem can be circumvented by measuring the R

[1] These curves were obtained with the method of D. Lee and W. A. Backofen, op. cit.
[2] W. A. Backofen, W. F. Hosford, Jr., and J. J. Burke, *ASM Trans Q.*, vol. 55, p. 264, 1962.

value, the ratio of the width strain to the thickness strain

$$R = \frac{\ln(w_0/w)}{\ln(t_0/t)} \tag{3-30}$$

Since $(Z/Y)^2 = \frac{1}{2}(1 + R)$, the equation or the yield locus can be written as

$$\sigma_1^2 + \sigma_2^2 - \frac{2R}{1+R}\sigma_1\sigma_2 = Y^2 \tag{3-31}$$

High through-thickness yield stress Z results in low-thickness strain and a high value of R. The extent of strengthening from the texture effect can be seen from Fig. 3-6. For a spherical pressure vessel $\sigma_1 = \sigma_2$. Thus, by moving out a 45° line on Fig. 3-6, we see that the resistance to yielding increases markedly with increased R.

3-8 YIELD SURFACE AND NORMALITY

The relationships that have been developed for yield criteria, Eqs. (3-12) and (3-21), can be represented geometrically by a cylinder oriented at equal angles to the $\sigma_1, \sigma_2, \sigma_3$ axes (Fig. 3-7). A state of stress which gives a point inside of the cylinder represents elastic behavior. Yielding begins when the state of stress reaches the surface of the cylinder, which is called the *yield surface*. The radius of the cylinder MN is the stress deviator. Since the axis of the cylinder OM makes equal angles with the principal stress axes, $l = m = n = 1/\sqrt{3}$, and from Eq. (2-18), $\sigma = (\sigma_1 + \sigma_2 + \sigma_3)/3 = \sigma_m$. Therefore, the axis of the cylinder is the hydrostatic component of stress. Since plastic deformation is not influenced by hydrostatic stress, the generator of the yield surface is a straight line parallel to OM, so that the radius of the cylinder is constant. As plastic deformation occurs we can consider that the yield surface expands outward, maintaining its same geometric shape.

The yield surface shown in Fig. 3-7 is a circular cylinder if its represents the von Mises' yield criterion. If a plane is passed through this surface parallel to the σ_2 axis, it intersects on the $\sigma_1\sigma_3$ plane as an ellipse (see Fig. 3-5). The yield surface

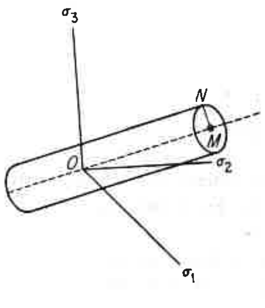


Figure 3-7 Yield surface for von Mises' criterion.

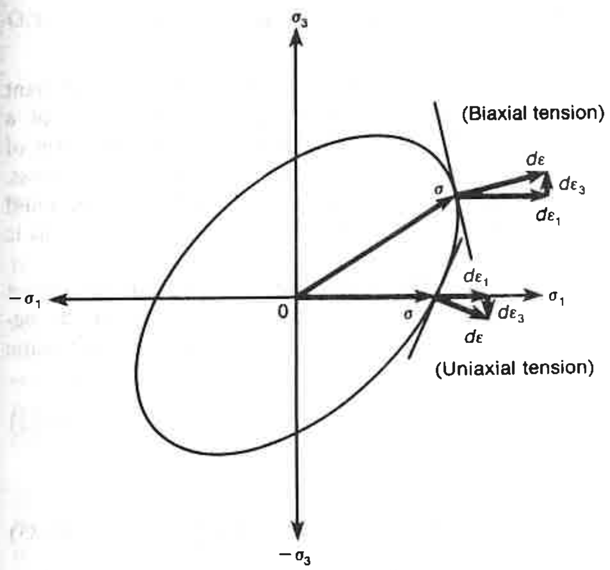


Figure 3-8 Example of the usefulness of the normality rule in working with the yield locus. Note the total strain vector $d\varepsilon$ is normal to the yield locus.

for the maximum-shear-stress criterion is a hexagonal cylinder. It should be noted that although the yield surface is an important concept in plasticity theory, there is no extensive body of experimental data on the shape of the surface. There is some work[1] which indicates that the yield surface is not a cylinder of uniform radius.

Drucker[2] has shown that the total plastic strain vector must be normal to the yield surface. As a consequence, any acceptable yield surface must be convex about its origin. Because of normality there is no component of the total strain vector that acts in the direction of σ_m. Therefore, the hydrostatic component of stress does not act to expand the yield surface. Because the deviatoric component of stress acts in the same direction as the total strain vector their dot product causes the plastic work as the yield surface is expanded by plastic deformation.

The normality rule also is useful in constructing experimental yield loci.[3] Figure 3-8 shows that the total strain vector $d\varepsilon$ is normal to the yield locus. We are looking at the projection of $d\varepsilon$ on the 1-3 plane. If the yield locus is known we can establish the ratio $d\varepsilon_1 : d\varepsilon_3$ from the normality rule. In the more usual case, $d\varepsilon_1/d\varepsilon_3$ is known experimentally and when combined with the normality condition they establish part of the yield locus.

[1] L. W. Hu, J. Markowitz, and T. A. Bartush, *Exp. Mech.* vol. 6, pp. 58–65, 1966.
[2] D. C. Drucker, Proceedings 1st U.S. National Congress of Applied Mechanics, p. 487, 1951.
[3] W. A. Backofen, "Deformation Processing," pp. 58–72, Addison-Wesley, Reading, Mass., 1972.

3-9 OCTAHEDRAL SHEAR STRESS AND SHEAR STRAIN

The octahedral stresses are a particular set of stress functions which are important in the theory of plasticity. They are the stresses acting on the faces of a three-dimensional octahedron which has the geometric property that the faces of the planes make equal angles with each of the three principal directions of stress. For such a geometric body, the angle between the normal to one of the faces and the nearest principal axis is 54°44′, and the cosine of this angle is $1/\sqrt{3}$. This is equivalent to {111} plane in an fcc crystal lattice.

The stress acting on each face of the octahedron can be resolved[1] into a normal octahedral stress σ_{oct} and an octahedral shear stress lying in the octahedral plane, τ_{oct}. The normal octahedral stress is equal to the hydrostatic component of the total stress.

$$\sigma_{oct} = \frac{\sigma_1 + \sigma_2 + \sigma_3}{3} = \sigma_m \tag{3-32}$$

The octahedral shear stress τ_{oct} is given by

$$\tau_{oct} = \tfrac{1}{3}\left[(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2\right]^{1/2} \tag{3-33}$$

Since the normal octahedral stress is a hydrostatic stress, it cannot produce yielding in solid materials. Therefore, the octahedral shear stress is the component of stress responsible for plastic deformation. In this respect, it is analogous to the stress deviator.

If it is assumed that a critical octahedral shear stress determines yielding, the failure criterion can be written as

$$\tau_{oct} = \tfrac{1}{3}\left[(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2\right]^{1/2} = \frac{\sqrt{2}}{3}\sigma_0$$

or

$$\sigma_0 = \frac{1}{\sqrt{2}}\left[(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2\right]^{1/2} \tag{3-34}$$

Since Eq. (3-34) is identical with the equation already derived for the distortion-energy theory, the two yielding theories give the same results. In a sense, the octahedral theory can be considered the *stress equivalent* of the distortion-energy theory. According to this theory, the octahedral shear stress corresponding to yielding in uniaxial stress is given by

$$\tau_{oct} = \frac{\sqrt{2}}{3}\sigma_0 = 0.471\sigma_0 \tag{3-35}$$

Octahedral strains are referred to the same three-dimensional octahedron as the octahedral stresses. The octahedral linear strain is given by

$$\varepsilon_{oct} = \frac{\varepsilon_1 + \varepsilon_2 + \varepsilon_3}{3} \tag{3-36}$$

[1] A. Nadai, "Theory of Flow and Fracture of Solids," 2d Ed., vol. I, pp. 99–105, McGraw-Hill Book Co., New York, 1950.

Octahedral shear strain is given by

$$\gamma_{oct} = \tfrac{2}{3}\left[(\varepsilon_1 - \varepsilon_2)^2 + (\varepsilon_2 - \varepsilon_3)^2 + (\varepsilon_3 - \varepsilon_1)^2\right]^{1/2} \tag{3-37}$$

3-10 INVARIANTS OF STRESS AND STRAIN

It is frequently useful to simplify the representation of a complex state of stress or strain by means of invariant functions of stress and strain. If the plastic stress-strain curve (the flow curve) is plotted in terms of invariants of stress and strain, approximately the same curve will be obtained regardless of the state of stress. For example, the flow curves obtained in a uniaxial-tension test and a biaxial-torsion test of a thin tube with internal pressure will coincide when the curves are plotted in terms of invariant stress and strain functions.

Nadai[1] has shown that the octahedral shear stress and shear strain are invariant functions which describe the flow curve independent of the type of test. However, the most frequently used invariant function to describe plastic deformation is *effective stress* $\bar{\sigma}$ or *effective strain* $\bar{\varepsilon}$.

$$\bar{\sigma} = \frac{\sqrt{2}}{2}\left[(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2\right]^{1/2} \tag{3-38}$$

$$d\bar{\varepsilon} = \frac{\sqrt{2}}{3}\left[(d\varepsilon_1 - d\varepsilon_2)^2 + (d\varepsilon_2 - d\varepsilon_3)^2 + (d\varepsilon_3 - d\varepsilon_1)^2\right]^{1/2} \tag{3-39}$$

The above equation for effective strain can be simplified as[2]

$$d\bar{\varepsilon} = \left[\tfrac{2}{3}\left(d\varepsilon_1^2 + d\varepsilon_2^2 + d\varepsilon_3^2\right)\right]^{1/2} \tag{3-40}$$

or in terms of total plastic strain

$$\bar{\varepsilon} = \left[\tfrac{2}{3}\left(\varepsilon_1^2 + \varepsilon_2^2 + \varepsilon_3^2\right)\right]^{1/2} \tag{3-41}$$

The strains used in Eqs. (3-39), (3-40), and (3-41) should be the plastic portion of the total strain. Frequently this is indicated by the notation ε_i^P, where $\varepsilon_i^P = \varepsilon_i(\text{total}) - \varepsilon_i(\text{elastic})$. In dealing with problems in metalworking the elastic strain is negligible, but in plasticity problems involving strains at a notch, overstressing of pressure vessels, etc., the elastic strains usually cannot be ignored.

Example Show that the equations for significant stress and strain reduce to the values for a tensile test.

[1] A. Nadai, *J. Appl. Phys.*, vol. 8, p. 205, 1937.
[2] W. E. Hosford and R. M. Caddell, "Metal Forming: Mechanics and Metallurgy," pp. 44–46, Prentice-Hall, Inc., Englewood Cliffs, N.J., 1983.

For a tensile test $\sigma_1 \neq 0$; $\sigma_2 = \sigma_3 = 0$, so from Eq. (3-38)

$$\bar{\sigma} = \frac{\sqrt{2}}{2}\left[\sigma_1^2 + \sigma_1^2\right] = \frac{\sqrt{2}\sqrt{2}}{2}\sigma_1 = \sigma_1$$

The strains in the tensile test are ε_1; $\varepsilon_2 = \varepsilon_3 \neq \varepsilon_1$ but from $\varepsilon_1 + \varepsilon_2 + \varepsilon_3 = 0$

$$\varepsilon_1 + 2\varepsilon_2 = 0 \quad \text{and} \quad d\varepsilon_1 = -2\,d\varepsilon_2 = -2\,d\varepsilon_3$$

$$d\bar{\varepsilon} = \left[\frac{2}{3}\left(d\varepsilon_1^2 + d\varepsilon_2^2 + d\varepsilon_3^2\right)\right]^{1/2} = \left[\frac{2}{3}\left(d\varepsilon_1^2 + \frac{d\varepsilon_1^2}{4} + \frac{d\varepsilon_1^2}{4}\right)\right]^{1/2}$$

$$d\bar{\varepsilon} = \left[\tfrac{2}{3}\left(\tfrac{6}{4}\right) d\varepsilon_1^2\right]^{1/2} = d\varepsilon_1$$

Thus the power law expression for the flow curve, Eq. (3-1) may be used as a first approximation to predict the plastic stress-strain behavior in other than tensile forms of loading.

$$\bar{\sigma} = K\bar{\varepsilon}^n \tag{3-42}$$

3-11 PLASTIC STRESS-STRAIN RELATIONS

Having discussed the relationships between stress state and plastic yielding, it is now necessary to consider the relations between stress and strain in plastic deformation. In the elastic region the strains are uniquely determined by the stresses through Hooke's law without regard to how the stress state was achieved. This is not the case for plastic deformation. In the plastic region the strains in general are not uniquely determined by the stresses but depend on the entire history of loading. Therefore, in plasticity it is necessary to determine the differentials or *increments of plastic strain* throughout the loading path and then obtain the total strain by integration or summation. As a simple example, consider a rod 50 mm long extended in tension to 60 mm and then compressed to the original 50 mm length.

On the basis of total deformation

$$\varepsilon = \int_{50}^{60} \frac{dL}{L} + \int_{60}^{50} \frac{dL}{L} = 0$$

However, on an incremental basis

$$\varepsilon = \int_{50}^{60} \frac{dL}{L} + \int_{60}^{50} -\frac{dL}{L} = 2 \ln 1.2 = 0.365$$

For the particular class of loading paths in which all the stresses increase in the same ratio, *proportional loading*, i.e.,

$$\frac{d\sigma_1}{\sigma_1} = \frac{d\sigma_2}{\sigma_2} = \frac{d\sigma_3}{\sigma_3}$$

the plastic strains are independent of the loading path and depend only on the final state of stress.

There are two general categories of plastic stress-strain relationships. *Incremental* or *flow theories* relate the stresses to the plastic strain increments. *Deformation* or *total strain theories* relate the stresses to the total plastic strain. Deformation theory simplifies the solution of plasticity problems, but the plastic strains in general cannot be considered independent of loading path.

Levy-Mises Equations (Ideal Plastic Solid)

The relationship between stress and strain for an ideal plastic solid, where the elastic strains are negligible, are called *flow rules* or the Levy-Mises equations. If we consider yielding under uniaxial tension, then $\sigma_1 \neq 0$, $\sigma_2 = \sigma_3 = 0$, and $\sigma_m = \sigma_1/3$. Since only the deviatoric stresses cause yielding

$$\sigma_1' = \sigma_1 - \sigma_m = \frac{2\sigma_1}{3}; \qquad \sigma_2' = \sigma_3' = \frac{-\sigma_1}{3}$$

from which we find

$$\sigma_1' = -2\sigma_2' = -2\sigma_3' \tag{3-43}$$

From the condition of constancy of volume in plastic deformation

$$d\varepsilon_1 = -2\,d\varepsilon_2 = -2\,d\varepsilon_3 \tag{3-44}$$

so that

$$\frac{d\varepsilon_1}{d\varepsilon_2} = -2 = \frac{\sigma_1'}{\sigma_2'} \tag{3-45}$$

This can be generalized to the Levy-Mises equation

$$\frac{d\varepsilon_1}{\sigma_1'} = \frac{d\varepsilon_2}{\sigma_2'} = \frac{d\varepsilon_3}{\sigma_3'} = d\lambda \tag{3-46}$$

These equations express the fact that at any instant of deformation the ratio of the plastic strain increments to the current deviatoric stresses is constant.

By using Eqs. (2-57) the above equations can be written in terms of the actual stresses.

$$d\varepsilon_1 = \tfrac{2}{3}\,d\lambda\left[\sigma_1 - \tfrac{1}{2}(\sigma_2 + \sigma_3)\right], \quad \text{etc.}$$

To evaluate $d\lambda$ we utilize the effective strain, Eq. (3-39), which yields $d\bar{\varepsilon} = \frac{2}{3}\,d\lambda\,\bar{\sigma}$.

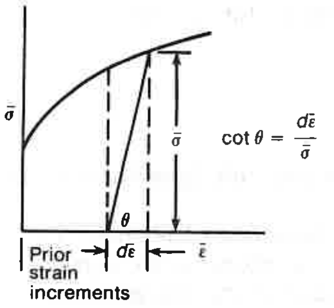


Figure 3-9 Method of establishing $d\bar{\varepsilon}/\bar{\sigma}$ in Eq. (3-47).

The Levy-Mises equations then become

$$
\begin{aligned}
d\varepsilon_1 &= \frac{d\bar{\varepsilon}}{\bar{\sigma}}\left[\sigma_1 - \frac{1}{2}(\sigma_2+\sigma_3)\right] \\
d\varepsilon_2 &= \frac{d\bar{\varepsilon}}{\bar{\sigma}}\left[\sigma_2 - \frac{1}{2}(\sigma_3+\sigma_1)\right] \\
d\varepsilon_3 &= \frac{d\bar{\varepsilon}}{\bar{\sigma}}\left[\sigma_3 - \frac{1}{2}(\sigma_1+\sigma_2)\right]
\end{aligned}
\tag{3-47}
$$

The similarity with Eqs. (2-64) for the elastic solid should be noted. In place of $1/E$ the flow rules have a ratio $d\bar{\varepsilon}/\bar{\sigma}$ which changes throughout the course of the deformation. In place of ν they have the value $\frac{1}{2}$. The proportionality constant $d\bar{\varepsilon}/\bar{\sigma}$ is evaluated from an effective stress-effective strain curve for an increment of plastic strain $d\bar{\varepsilon}$ in the manner shown in Fig. 3-9.

Example An aluminum thin-walled tube (radius/thickness = 20) is closed at each end and pressurized to 7 MPa to cause plastic deformation. Neglect the elastic strain and find the plastic strain in the circumferential (hoop) direction of the tube. The plastic stress-strain curve is given by $\bar{\sigma} = 170(\bar{\varepsilon})^{0.25}$, where stress is in MPa.

From the strength of materials equations for thin-walled pressure vessels, the stresses on the outside of the tube are:

$$\sigma_\theta = \sigma_1 = \frac{pr}{t} \quad \text{(circumferential direction)}$$

$$\sigma_l = \sigma_2 = \frac{pr}{2t} = \frac{\sigma_1}{2} \quad \text{(longitudinal direction)}$$

$$\sigma_r = \sigma_3 = 0 \quad \text{(radial direction)}$$

From the Levy-Mises equations

$$d\varepsilon_1 = \frac{d\bar{\varepsilon}}{\bar{\sigma}}\left[\sigma_1 - \frac{1}{2}(\sigma_2+\sigma_3)\right] = \frac{d\bar{\varepsilon}}{\bar{\sigma}}\left[\sigma_1 - \frac{\sigma_1}{4}\right] = \frac{d\bar{\varepsilon}}{\bar{\sigma}}\left(\frac{3\sigma_1}{4}\right)$$

$$d\varepsilon_3 = \frac{d\bar{\varepsilon}}{\bar{\sigma}}\left[\sigma_3 - \frac{1}{2}(\sigma_1+\sigma_2)\right] = \frac{d\bar{\varepsilon}}{\bar{\sigma}}\left[0 - \frac{3\sigma_1}{4}\right]$$

$$\therefore \quad d\varepsilon_1 = -d\varepsilon_3 \quad \text{and from} \quad d\varepsilon_1 + d\varepsilon_2 + d\varepsilon_3 = 0 \qquad d\varepsilon_2 = 0$$

$$\bar{\sigma} = \frac{1}{\sqrt{2}}\left[\left(\sigma_1 - \frac{\sigma_1}{2}\right)^2 + \left(\frac{\sigma_1}{2} - 0\right)^2 + (0-\sigma_1)^2\right]^{1/2}$$

$$= \frac{1}{\sqrt{2}}\left[\frac{6}{4}\sigma_1\right]^2 = \frac{\sqrt{3}}{2}\sigma_1$$

$$\sigma_1 = \frac{pr}{t} = 7(20) = 140 \text{ MPa} \qquad \bar{\sigma} = \frac{\sqrt{3}}{2}(140) = 121 \text{ MPa}$$

$$\bar{\sigma} = 170(\bar{\varepsilon})^{0.25} \qquad \bar{\varepsilon} = \left(\frac{121}{170}\right)^{1/0.25} = (0.712)^4 = 0.257$$

$$d\bar{\varepsilon} = \frac{\sqrt{2}}{3}[(d\varepsilon_1 - 0)^2 + (0-(-d\varepsilon_1))^2 + (-d\varepsilon_1 - d\varepsilon_1)^2]^{1/2}$$

$$d\bar{\varepsilon} = \frac{\sqrt{2}}{3}\sqrt{6}\,d\varepsilon_1 = \frac{2}{\sqrt{3}}d\varepsilon_1$$

$$d\varepsilon_1 = \frac{\sqrt{3}}{2}d\bar{\varepsilon} \qquad \varepsilon_1 = \frac{\sqrt{3}}{2}\int_0^{\bar{\varepsilon}} d\bar{\varepsilon} = \frac{\sqrt{3}}{2}(\bar{\varepsilon}) = \frac{\sqrt{3}}{2}(0.257) = 0.222$$

Prandtl-Reuss Equations (Elastic-Plastic Solid)

The Levy-Mises equations can only be applied to problems of large plastic deformation because they neglect elastic strains. To treat the important, but more difficult problems in the elastic-plastic region it is necessary to consider both elastic and plastic components of strain. These equations were proposed by Prandtl (1925) and Reuss (1930).

The total strain increment is the sum of an elastic strain increment de^E and a plastic strain increment $d\varepsilon^P$.

$$d\varepsilon_{ij} = de^E_{ij} + d\varepsilon^P_{ij} \tag{3-48}$$

From Eqs. (2-52) and (2-69), the elastic strain increment is given by

$$de^E_{ij} = \left(de_{ij} - \frac{de_{kk}}{3}\delta_{ij}\right) + \frac{de_{kk}}{3}\delta_{ij} = \frac{1+\nu}{E}d\sigma_{ij} - \frac{\nu}{E}\sigma_{kk}\delta_{ij}$$

or

$$de^E_{ij} = \frac{1+\nu}{E}d\sigma'_{ij} + \frac{1-2\nu}{E}\frac{d\sigma_{kk}}{3}\delta_{ij} \tag{3-49}$$

The plastic strain increment is given by the Levy-Mises equations, which can be written as

$$d\varepsilon_{ij}^{P} = \frac{3}{2}\frac{d\bar{\varepsilon}}{\bar{\sigma}}\sigma'_{ij} \tag{3-50}$$

Thus, the stress, strain relations for an elastic-plastic solid are given by

$$d\varepsilon_{ij} = \frac{1+\nu}{E}\,d\sigma'_{ij} + \frac{1+2\nu}{E}\frac{d\sigma_{kk}}{3}\delta_{ij} + \frac{3}{2}\frac{d\bar{\varepsilon}}{\bar{\sigma}}\sigma'_{ij} \tag{3-51}$$

Solution of Plasticity Problems

The Levy-Mises and Prandtl-Reuss equations provide relations between the increments of plastic strain and the stresses. The basic problem is to calculate the next increment of plastic strain for a given state of stress when the loads are increased incrementally. If all of the increments of strain are known, then the total plastic strain is simply determined by summation. To do this we have available a set of plastic stress-strain relationships, either Eqs. (3-47) or (3-51), a yield criterion, and a basic relationship for the flow behavior of the material in terms of a curve of $\bar{\sigma}$ vs. $\bar{\varepsilon}$. In addition, a complete solution also must satisfy the equations of equilibrium, the strain-displacement relations, and the boundary conditions. The reader is referred to the several excellent texts on plasticity listed at the end of this chapter for examples of detailed solutions.[1] Although the incremental nature of plasticity solutions in the past has resulted in much labor and infrequent application of the available techniques, the current widespread use of digital computers and finite element analysis should make plasticity analysis of engineering problems more commonplace.

3-12 TWO-DIMENSIONAL PLASTIC FLOW—SLIP-LINE FIELD THEORY

In many practical problems, such as rolling and strip drawing, all displacements can be considered to be limited to the xy plane, so that strains in the z direction can be neglected in the analysis. This is known as a condition of *plane strain*. When a problem is too difficult to an exact three-dimensional solution, a good indication of the stresses often can be obtained by consideration of the analogous plane-strain problem.

Since a plastic material tends to deform in all directions, to develop a plane-strain condition it is necessary to constrain flow in one direction. Constraint can be produced by an external lubricated barrier, such as a die wall (Fig. 3-10a), or it can arise from a situation where only part of the material is deformed and the rigid (elastic) material outside the plastic region prevents the spread of deformation (Fig. 3-10b).

[1] A number of plasticity problems are worked out in great detail in Lubahn and Felgar op. cit., Chaps 8 and 9.

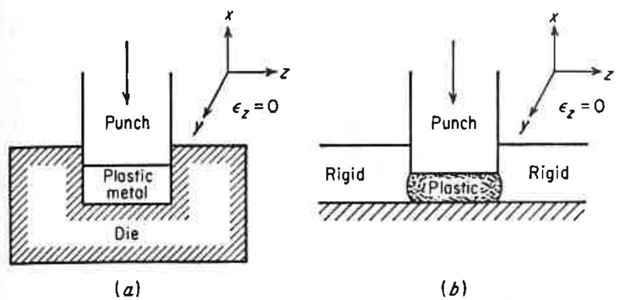


Figure 3-10 Methods of developing plastic constraint.

If the plane-strain deformation occurs on planes parallel to the xy plane, then

$$\varepsilon_z = \varepsilon_{xz} = \varepsilon_{yz} = 0 \quad \text{and} \quad \tau_{xz} = \tau_{yz} = 0$$

Since $\tau_{xz} = \tau_{yz} = 0$, it follows that σ_z is a principal stress. From the Levy-Mises equations, Eq. (3-47)

$$d\varepsilon_z = 0 = \frac{d\bar{\varepsilon}}{\bar{\sigma}}\left[\sigma_z - \frac{1}{2}(\sigma_x + \sigma_y)\right]$$

and

$$\sigma_z = \frac{\sigma_x + \sigma_y}{2} \tag{3-52}$$

Note that although the strain is zero in the z direction, a restraining stress acts in this direction.

Equation (3-52) could just as well have been written in terms of the principal stresses $\sigma_3 = (\sigma_1 + \sigma_2)/2$.

This principal stress will be intermediate between σ_1 and σ_2, so that the maximum-shear-stress yield criterion is given by

$$\sigma_1 - \sigma_2 = \sigma_0 = 2k \tag{3-53}$$

where k is the yield stress in pure shear.

If the value for the intermediate principal stress σ_3 is substituted into the von Mises' yield criterion, Eq. (3-12) it reduces to

$$\sigma_1 - \sigma_2 = \frac{2}{\sqrt{3}}\sigma_0 \tag{3-54}$$

However, for the von Mises' yield criterion $\sigma_0 = \sqrt{3}\,k$ so that Eq. (3-54) becomes

$$\sigma_1 - \sigma_2 = 2k \tag{3-55}$$

Thus, for a state of plane strain the maximum-shear stress and von Mises' yield criteria are equivalent. It can be considered that two-dimensional plastic flow will begin when the shear stress reaches a critical value of k.

Slip-line field theory is based on the fact that any general state of stress in plane strain consists of *pure shear* plus a *hydrostatic pressure*. We could show this

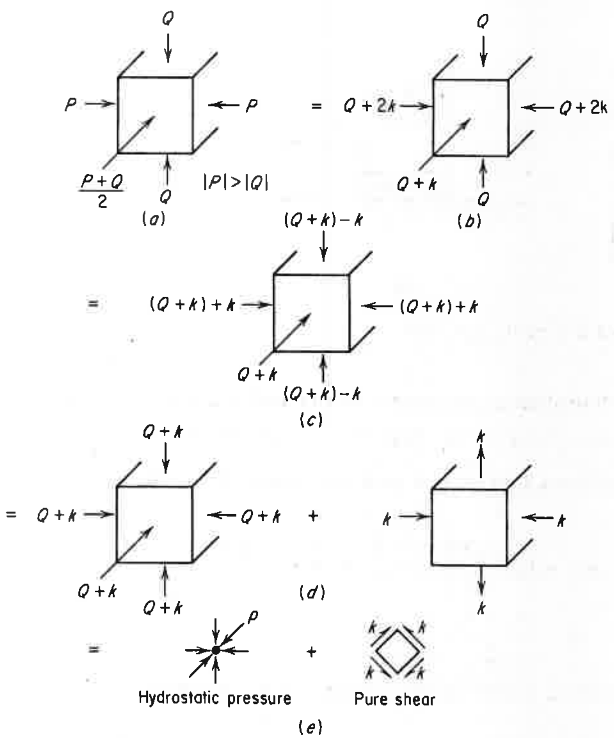


Figure 3-11 Demonstration that a state of stress in plane strain may be expressed as the sum of a hydrostatic stress and pure shear.

by applying the equations for transformation of stress from one set of axes to another, Eqs. (2-5) to (2-7), but it perhaps is more instructive to see this diagrammatically. In Fig. 3-11, let the state of stress consist of $\sigma_1 = -Q$, $\sigma_3 = -P$, and $\sigma_2 = (-P - Q)/2$. The maximum shear stress is given by

$$\tau_{max} = \sigma_1 - \sigma_3 = 2k$$

$$-Q + P = 2k$$

or in Fig. 3-11*b*,

$$P = Q + 2k$$

But we can write the state of stress in Fig. 3-11*b* as in Fig. 3-11*c*, which in turn can be written as the sum of a hydrostatic pressure and a biaxial state of stress Fig. 3-11*d*. The latter is the stress state in pure torsion, which for planes rotated by 45° consists of pure shear stresses. Thus, a general state of stress in plane strain can be decomposed into a hydrostatic state of stress p (in this case compression) and a state of pure shear k. The components of the stress tensor for plane strain are

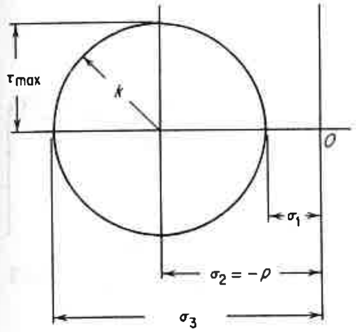


Figure 3-12 Mohr's circle representation of stresses in Fig. 3-9*a*.

$$\sigma_{ij} = \begin{vmatrix} p & k & 0 \\ k & p & 0 \\ 0 & 0 & p \end{vmatrix}$$

Mohr's circle representation for the state of stress given in Fig. 3-11 is shown in Fig. 3-12. If $\sigma_1 = -Q$ and $\sigma_3 = -P$, then $\sigma_2 = (-Q - P)/2 = -p$. This follows because

$$p = \sigma_m = \frac{\sigma_1 + \sigma_2 + \sigma_3}{3} = -\frac{1}{3}\left(Q + \frac{Q}{2} + \frac{P}{2} + P\right)$$

$$\therefore \quad p = \frac{Q + P}{2} = -\sigma_2$$

Also, the radius of Mohr's circle is $\tau_{max} = k$, where k is the yield stress in pure shear. Thus, using Fig. 3-12, we can express the principal stresses

$$\sigma_1 = -p + k$$
$$\sigma_2 = -p$$
$$\sigma_3 = -p - k$$

The slip-line field theory for plane strain allows the determination of stresses in a plastically deformed body when the deformation is not uniform throughout the body. In addition to requiring plane-strain conditions, the theory assumes an isotropic, homogeneous, rigid ideal plastic material. For such a non-strain-hardening material k is everywhere constant but p may vary from point to point. The state of stress at any point can be determined if we can find the magnitude of p and the direction of k. The lines of maximum shear stress occur in two orthogonal directions α and β. These lines of maximum shear stress are called *slip lines* and have the property that shear strain is a maximum and linear strain is zero tangent to their directions. The slip lines give the direction of p at any point and the changes in magnitude of p are deduced from the rotation of the slip line

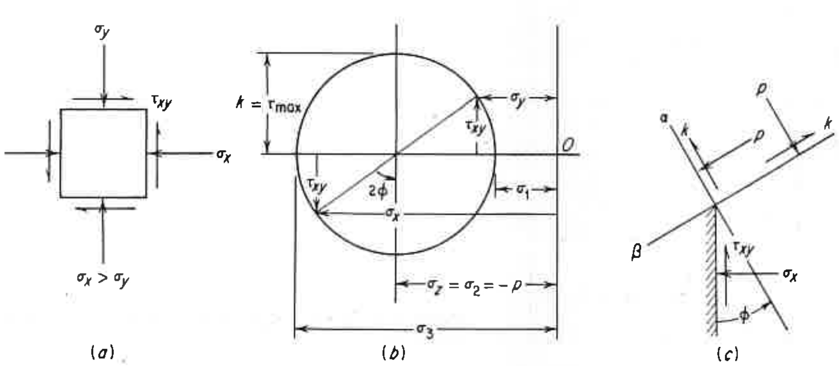


Figure 3-13 (a) Stress state on physical body; (b) Mohr's circle for (a); (c) relationship of physical body and α and β slip lines.

between one point and another in the field. It should be noted that the slip lines referred to in this section are geometric constructions which define the characteristic directions of the hyperbolic partial differential equations for the stress under plane-strain conditions. These slip lines bear no relationship to the slip lines observed under the microscope on the surface of a plastically deformed metal.

To arrive at the equations for calculating stress through the use of slip-line fields, we must now relate the stresses on a physical body in the xy coordinate system to p and k. Figure 3-13b shows the Mohr's circle representation of the stress state given in Fig. 3-13a. The stresses may be expressed as

$$\sigma_x = -p - k\sin 2\phi$$
$$\sigma_y = -p - (-k\sin 2\phi) = -p + k\sin 2\phi$$
$$\sigma_z = -p$$
$$\tau_{xy} = k\cos 2\phi$$

where 2ϕ is a counterclockwise angle on Mohr's circle from the *physical x* plane to the first plane of maximum shear stress. This plane of maximum shear stress is known as an α slip line. The relationship between the stress state on the physical body and the α and β slip lines is given in Fig. 3-13c.

The variation of hydrostatic pressure p with change in direction of the slip lines is given by the *Hencky equations*

$$p + 2k\phi = \text{constant along an } \alpha \text{ line} \qquad (3\text{-}56)$$
$$p - 2k\phi = \text{constant along a } \beta \text{ line}$$

These equations are developed[1] from the equilibrium equations in plane strain. The use of the Hencky equations will be illustrated with the example of the

[1] See for example W. Johnson and P. B. Mellor, "Plasticity for Mechanical Engineers," pp. 263–265, D. Van Nostrand Company, Inc., Princeton, N.J., 1962.

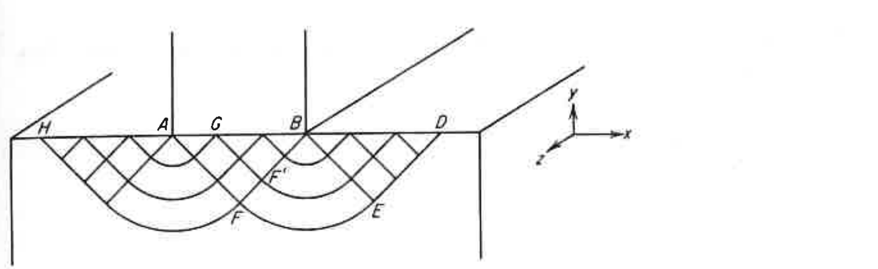


Figure 3-14 Slip-line field for frictionless indentation with a flat punch.

indentation of a thick block with a flat frictionless punch. The slip-line field shown in Fig. 3-14 was first suggested by Prandtl[1] in 1920. At the free surface on the frictionless interface between the punch and the block the slip lines meet the surface at 45° (see Prob. 3-15). We could construct the slip-line field by starting with triangle AFB, but we would soon see that if all plastic deformation were restricted to this region, the metal could not move because it would be surrounded by rigid (elastic) material. Therefore, the plastic zone described by the slip-line field must be extended along the free surface to AH and BD.

To determine the stresses from the slip-line field, we start with a simple point such as D. Since D is on a free surface, there is no stress normal to this surface.

$$\sigma_y = 0 = -p + k\sin 2\phi$$

and
$$\sigma_x = -p - k\sin 2\phi = -p - p = -2p$$

The stresses at point D are shown in Fig. 3-15. From the Mohr's circle we learn that $p = k$. In order to use the Hencky equations we need to know whether the slip line through D is an α or β line. This is done most simply from the following sign convention:

> For a counterclockwise rotation about the point of intersection of two slip lines, starting from an α-line the direction of the algebraically highest principal stress σ_1 is crossed before a β line is crossed.

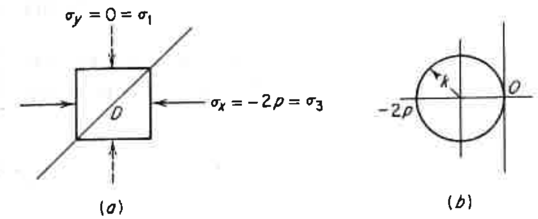


Figure 3-15 (a) Stresses at point D; (b) Mohr's circle.

[1] A different slip-line field was later suggested by R. Hill. Although the slip field is different, it leads to the same value of indentation pressure. This illustrates the fact that slip-line field solutions are not necessarily unique.

Applying this convention, we see that the slip line from D to E is an α line. Thus, the first Hencky equation applies,

$$p + 2k\phi = C_1$$

and if we use DE as the reference direction so $\phi = 0$,

$$p = C_1 = k$$

Because DE is straight p is constant from D to E and

$$p_D = p_E = k$$

Between E and F the tangent to the α slip line rotates through $\pi/2$ rad. Since the tangent to the α line rotates *clockwise*, $d\phi = -\pi/2$. If we write the Hencky equation in differential form, for clarity

$$dp + 2k\,d\phi = 0$$

or

$$(p_F - p_E) + 2k(\phi_F - \phi_E) = 0$$

$$p_F - k + 2k\left(-\frac{\pi}{2} - 0\right) = 0$$

$$p_F = k(\pi + 1)$$

Note that the pressure at F' is the same as at F because the slip line is straight and that the value of p under the punch face at G is also the same. (We stayed away from A and B at the punch edges because these are points of pressure discontinuity.) To find the punch pressure required to indent the block, it is necessary to convert the hydrostatic pressure at the punch interface into the vertical stress σ_y.

$$p_F = p_{F'} = p_G = k(\pi + 1)$$

$$\sigma_y = -p_G + k \sin 2\phi$$

From Fig. 3-13*c*, recall that the angle ϕ is measured by the counterclockwise angle from the physical x axis to the α line.

$$\sigma_y = -k(\pi + 1) + k \sin 2\left(\frac{3\pi}{4}\right)$$

$$\sigma_y = -k\pi - k - k = -2k\left(1 + \frac{\pi}{2}\right) \tag{3-57}$$

If we trace out other slip lines, we shall find in the same way that the normal compressive stress under the punch is $2k(1 + \pi/2)$, and the pressure is uniform.

Since $k = \sigma_0/\sqrt{3}$,

$$\sigma_y = \frac{2\sigma_0}{\sqrt{3}}\left(1 + \frac{\pi}{2}\right) \approx 3\sigma_0 \tag{3-58}$$

This shows that the yield pressure for the indentation of a thick block with a narrow punch is nearly three times the stress required for the yielding of a cylinder in frictionless compression. This increase in flow stress is a *geometrical constraint* resulting from the localized deformation under the narrow punch.

The example described above is one of the simplest situations that involves slip-line fields. In the general case the slip-line field selection must also satisfy certain velocity conditions to assure equilibrium. Prager[1] and Thomsen[2] have given general procedures for constructing slip-line fields. However, there is no easy method of checking the validity of a solution. Partial experimental verification of theoretically determined slip-line fields has been obtained for mild steel by etching techniques[3] which delineate the plastically deformed regions. Highly localized plastic regions can be delineated by an etching technique in Fe-3% Si steel.[4]

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